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Type II ancient compact solutions to the Yamabe flow

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Abstract. We construct new type II ancient compact solutions to the Yamabe flow. Our solutions are rotationally symmetric and converge, as $t \to -\infty$, to a tower of two spheres. Their curvature operator changes sign. We allow two time-dependent parameters in our ansatz. We use perturbation theory, via fixed point arguments, based on sharp estimates on ancient solutions of the approximated linear equation and careful estimation of the error terms which allow us to make the right choice of parameters. Our technique may be viewed as a parabolic analogue of gluing two exact solutions to the rescaled equation, that is the spheres, with narrow cylindrical necks to obtain a new ancient solution to the Yamabe flow. The result generalizes to the gluing of k spheres for any $k \geq 2$, in such a way the configuration of radii of the spheres glued is driven as $t \to -\infty$ by a *First order Toda system*.

1. Introduction

Let (M, g_0) be a compact manifold without boundary of dimension $n \geq 3$. If $g = v^{\frac{4}{n-2}} g_0$ is a metric conformal to g_0, the scalar curvature R of g is given in terms of the scalar curvature R_0 of g_0 by

$$R = v^{-\frac{n+2}{n-2}}(-\bar{c}_n \Delta_{g_0} v + R_0 v)$$

where Δ_{g_0} denotes the Laplace Beltrami operator with respect to g_0 and $\bar{c}_n = 4(n-1)/(n-2)$.

In 1989 R. Hamilton introduced the *Yamabe flow*

$$\frac{\partial g}{\partial t} = -Rg \tag{1.1}$$

as an approach to solve the *Yamabe problem* on manifolds of positive conformal Yamabe invariant. It is the negative L^2-gradient flow of the total scalar curvature, restricted to a given conformal class. The flow may be interpreted as deforming a Riemannian metric to a conformal metric of constant scalar curvature, when this flow converges.

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Hamilton [8] showed the existence of the normalized Yamabe flow (which is the reparametrization of (1.1) to keep the volume fixed) for all time; moreover, in the case when the scalar curvature of the initial metric is negative, he showed the exponential convergence of the flow to a metric of constant scalar curvature.

Since then, there has been a number of works on the convergence of the Yamabe flow on a compact manifold to a metric of constant scalar curvature. Chow [3] showed the convergence of the flow, under the conditions that the initial metric is locally conformally flat and of positive Ricci curvature. The convergence of the flow for any locally conformally flat initially metric was shown by Ye [20].

More recently, Schwetlick and Struwe [19] obtained the convergence of the Yamabe flow on a general compact manifold under a suitable Kazdan–Warner type of condition that rules out the formation of bubbles and that is verified (via the Positive Mass Theorem) in dimensions $3 \le n \le 5$. The convergence result, in its full generality, was established by Brendle [1, 2] (up to a technical assumption, in dimensions $n \ge 6$, on the rate of vanishing of Weyl tensor at the points at which it vanishes): starting with any smooth metric on a compact manifold, the normalized Yamabe flow converges to a metric of constant scalar curvature.

In the special case where the background manifold M_0 is the sphere S^n and g_0 is the standard spherical metric g_{S^n}, the Yamabe flow evolving a metric $g = v^{\frac{4}{n-2}}(\cdot,t)g_{S^n}$ takes (after rescaling in time by a constant) the form of the *fast diffusion equation*

$$(1.2)\qquad (v^{\frac{n+2}{n-2}})_t = \Delta_{S^n} v - c_n v, \quad c_n = \frac{n(n-2)}{4}.$$

Starting with any smooth metric g_0 on S^n, it follows by the results in [3], [20] and [6] that the solution of (1.2) with initial data g_0 will become singular at some finite time $t < T$ and v becomes spherical at time T, which means that after a normalization, the normalized flow converges to the spherical metric. In addition, v becomes extinct at T.

A metric $g = v^{\frac{4}{n-2}} g_{S^n}$ may also be expressed as a metric on $\mathbb{R}^n$ via stereographic projection. It follows that if $g = \bar{v}^{\frac{4}{n-2}}(\cdot,t)g_{\mathbb{R}^n}$ (where $g_{\mathbb{R}^n}$ denotes the standard metric on $\mathbb{R}^n$) evolves by the Yamabe flow (1.1), then $\bar{v}$ satisfies (after a rescaling in time) the fast diffusion equation on $\mathbb{R}^n$

$$(1.3)\qquad (\bar{v}^p)_t = \Delta \bar{v}, \quad p := \frac{n+2}{n-2}.$$

Observe that if $g = \bar{v}^{\frac{4}{n-2}}(\cdot,t)g_{\mathbb{R}^n}$ represents a smooth solution when lifted on S^n, then $\bar{v}(\cdot,t)$ satisfies the growth condition

$$\bar{v}(y,t) = O(|y|^{-(n-2)}), \quad \text{as } |y| \to \infty.$$

Definition 1.1. The solution $g = v(\cdot,t)^{\frac{4}{n-2}} g_0$ to (1.1) is called ancient if it exists for all time $t \in (-\infty, T)$, where $T < \infty$. We will say that the ancient solution g is of type I if it satisfies

$$\limsup_{t\to-\infty} \Big(|t| \max_{M_0} |\mathrm{Rm}|(\cdot,t) \Big) < \infty$$

(where Rm is the Riemannian curvature of metric $g = v(\cdot,t)^{\frac{4}{n-2}} g_0$ and can be expressed in terms of v and its first and second derivatives). An ancient solution which is not of type I will be called of type II.

Explicit examples of ancient solutions to the Yamabe flow on S^n are as follows.

Contracting spheres. They are special solutions v of (1.2) which depend only on time t and satisfy the ODE

$$\frac{dv^{\frac{n+2}{n-2}}}{dt} = -c_n v.$$

They are given by

$$v_S(p,t) = \left(\frac{4}{n+2} c_n (T-t)\right)^{\frac{n-2}{4}} \tag{1.4}$$

and represent a sequence of round spheres shrinking to a point at time $t = T$. They are shrinking solitons and type I ancient solutions.

King solutions. They were discovered by J. R. King [12]. They can be expressed on $\mathbb{R}^n$ in closed from, namely $g = \bar{v}_K(\cdot, t)^{\frac{4}{n-2}} g_{\mathbb{R}^n}$, where $\bar{v}_K$ is the radial function

$$\bar{v}_K(r,t) = \left(\frac{A(t)}{1 + 2B(t)r^2 + r^4}\right)^{\frac{n-2}{4}} \tag{1.5}$$

and the coefficients $A(t)$ and $B(t)$ satisfy a certain system of ODEs. The King solutions are *not solitons* and may be visualized, as $t \to -\infty$, as two Barenblatt self-similar solutions "glued" together to form a compact solution to the Yamabe flow. They are type I ancient solutions.

Let us make the analogy with the Ricci flow on S^2. The two explicit compact ancient solutions to the two-dimensional Ricci flow are the contracting spheres and the King–Rosenau solutions [12,13,17]. The latter ones are the analogues of the King solution (1.5) to the Yamabe flow. The difference is that the King–Rosenau solutions are type II ancient solutions to the Ricci flow while the King solution above is of type I.

It has been showed by Daskalopoulos, Hamilton and Sesum [4] that the spheres and the King–Rosenau solutions are the only compact ancient solutions to the two-dimensional Ricci flow. The natural question to raise is whether the analogous statement holds true for the Yamabe flow, that is, whether the contracting spheres and the King solution are the only compact ancient solutions to the Yamabe flow. This occurs not to be the case as the following discussion shows.

In this article we will construct ancient radially symmetric solutions of the Yamabe flow (1.2) on S^n other than the contracting spheres (1.4) and the King solutions (1.5). Our new solutions, as $t \to -\infty$, may be visualized as two spheres joint by a short neck. Their curvature operator changes sign and they are type II ancient solutions.

Before we present the ansatz of our construction we will perform a change of variables switching to cylindrical coordinates. Let $g = \bar{v}^{\frac{4}{n-2}}(\cdot, t) g_{\mathbb{R}^n}$ be a radially symmetric solution of (1.3) which becomes extinct at time T, namely $\bar{v} = \bar{v}(r,t)$ is a radial function on $\mathbb{R}^n$ that vanishes at T. One may introduce the cylindrical change of variables

$$u(x,\tau) = (T-t)^{-\frac{1}{p-1}} r^{\frac{2}{p-1}} \bar{v}(r,t), \quad r = e^x, \quad t = T(1 - e^{-\tau}).$$

In this language equation (1.3) becomes

$$(u^p)_\tau = u_{xx} + \alpha u^p - \beta u, \quad \beta = \frac{(n-2)^2}{4}, \quad \alpha = \frac{p}{p-1} = \frac{n+2}{4}. \tag{1.6}$$

From now on we will denote the time τ by t. By suitable scaling we can make the two constants α and β in (1.6) equal to 1, so that from now on we will consider the equation

$$(u^p)_t = u_{xx} + u^p - u. \tag{1.7}$$

The steady states of equation (1.7), namely the solutions w of the equation

$$w_{xx} + w^p - w = 0, \quad w(\pm\infty) = 0 \tag{1.8}$$

are given in closed form

$$w(x) = \left(\frac{k_n \lambda e^{\gamma x}}{1 + \lambda^2 e^{2\gamma x}}\right)^{\frac{n-2}{2}} = (2k_n \operatorname{sech}(\gamma x + \log\lambda))^{\frac{n-2}{2}} \tag{1.9}$$

with

$$\gamma = \frac{1}{\sqrt{\beta}} = \frac{2}{n-2} \quad \text{and} \quad k_n = \left(\frac{4n}{n-2}\right)^{\frac{1}{2}}.$$

It is known that $w(x)$ is the only even, positive solution of (1.8), given in cylindrical coordinates, after stereographic projection, geometrically representing the conformal metric for a sphere. Observe that

$$w(x) = O(e^{-|x|}), \quad \text{as } |x| \to \infty. \tag{1.10}$$

We will construct new evolving *ancient* compact metrics which look, for t close to $-\infty$, like two spheres glued by a narrow neck. We choose our *ansatz* for an ancient solution $u(x,t)$ of (1.7) to be of the form

$$u(x,t) = (1 + \eta(t)) z(x,t) + \psi(x,t) \tag{1.11}$$

with

$$z(x,t) = w(x + \xi(t)) + w(x - \xi(t)) \tag{1.12}$$

for suitable parameter functions $\eta(t), \xi(t)$. The perturbation function $\psi(x,t)$ will converge to zero, as $t \to -\infty$, in a suitable norm that will be defined below. More precisely,

$$\xi(t) = \xi_0(t) + h(t)$$

for a suitable parameter function $h(t)$. Both parameter functions $h(t)$ and $\eta(t)$ will decay in $|t|$, as $t \to -\infty$. Let

$$\xi_0(t) := \frac{1}{2}\log(2b|t|)$$

be a solution to

$$\dot{\xi} + be^{-2\xi} = 0, \quad \text{with} \quad b := \frac{\int_0^\infty w^p e^{-x}\,dx}{p\int_{\mathbb{R}} w'^2 w^{p-1}\,dx},$$

which is the homogeneous part of the nonhomogeneous equation (5.8). As we will explain below, equation (5.8) is derived as a consequence of adjusting parameters $h(t)$ and $\eta(t)$ so that our solution ψ satisfies the orthogonality conditions (3.5) and (3.6).

The main result in this article states as follows.

Theorem 1.1. *There exists a number t_0 and a solution $u(x,t)$ to* (1.7) *defined on $\mathbb{R} \times (-\infty, t_0]$, of the form* (1.11)–(1.12)*, with $\xi := \frac{1}{2}\log(2b|t|) + h(t)$, such that the functions $\psi(x,t)$, $\eta(t)$ and $h(t)$ tend to zero in appropriate norms as $t \to -\infty$. Moreover, u defines a radially symmetric ancient solution to the Yamabe flow* (1.1) *on S^n which is of type II (in the sense of Definition* 1.1*) and its Ricci curvature changes its sign.*

Theorem 1.1 shows that the classification of ancient solutions to the compact Yamabe flow on S^n poses a rather difficult, even maybe impossible task. On the other hand, it gives a new way of constructing ancient solutions. It shows how one may glue two ancient solutions of a parabolic equation, in our case of equation (1.7), to construct a new ancient solution to the same equation. This parabolic gluing becomes more and more apparent as $t \to -\infty$, since as $t \to +\infty$ it is known that our conformal factor approaches the one of the standard sphere.

Our construction can be generalized to give ancient solutions which, as $t \to -\infty$, may be visualized as a tower of n spheres joined by short necks. We refer to them as *moving towers of bubbles*. In terms of equation (1.7), for a given $k \geq 2$ we look for a solution of (1.7) of the form

$$u(x,t) = \sum_{j=1}^{k}(1+\eta_j(t))w(x-\xi_j(t)) + \psi(x,t) \tag{1.13}$$

where the functions ξ_j are ordered and symmetrically arranged,

$$\xi_1(t) < \xi_2(t) < \cdots < \xi_k(t), \quad \xi_j(t) = -\xi_{k-j+1}(t), \quad j = 0, \dots, k. \tag{1.14}$$

We have the following result.

Theorem 1.2. *Given $k \geq 2$, there exists a number t_0 and a solution $u(x,t)$ to* (1.7) *defined on $\mathbb{R} \times (-\infty, t_0]$, of the form* (1.13)–(1.14)*, with*

$$\xi_j(t) = \xi_{0j}(t) + h_j(t), \quad \xi_{0j}(t) = \left(j - \frac{k+1}{2}\right)\log(b|t|) + \gamma_j \tag{1.15}$$

for certain explicit constants γ_j, where the functions $\psi(x,t)$, $\eta_j(t)$ and $h_j(t)$ tend to zero in appropriate norms as $t \to -\infty$.

The functions ξ_{0j} in the above statement solve the *first order Toda system*

$$b^{-1}\dot{\xi}_j(t) + e^{-(\xi_{j+1}-\xi_j)} - e^{-(\xi_j-\xi_{j-1})} = 0, \quad j = 1, \dots, k,\ t \in (-\infty, -t_0], \tag{1.16}$$

with the conventions

$$\xi_0 \equiv -\infty, \quad \xi_{k+1} \equiv +\infty.$$

We will analyze this system in the last section.

Gluing techniques relying on linearization and perturbation theory have been used in many elliptic settings. We refer to the works of Kapouleas [9–11] on the gluing of two constant mean curvature surfaces to produce another constant mean curvature surface, and to the works on the works [14], [15] and [18] on the gluing of manifolds of constant scalar curvature to produce another manifold of constant scalar curvature. Gluing techniques have been used to construct new solutions to elliptic semilinear equations in [5] and [7]. Embedded self similar solutions of the mean curvature flow have been constructed in [16] by using gluing techniques. We use such techniques here in the parabolic setting as well. We expect that our way of constructing new ancient solutions to the Yamabe flow could be adopted to other geometric flows as well.

In the rest of the paper we will carry out in detail the proof of Theorem 1.1 and indicate in the last section the changes needed for the proof of the more general statement Theorem 1.2.

We will next indicate the main steps in proving Theorem 1.1.

(1) We first define the Banach space, which our ancient solution u to (1.7) belongs to and its associated norm. We also specify the spaces for our parameter functions $\eta(t)$ and $h(t)$ and their associated norms.

(2) Using the ansatz (1.11)–(1.12) for our solution u, we show that the perturbation term ψ is a solution to the equation

$$pz^{p-1}\partial_t\psi = \psi_{xx} - \psi + pz^{p-1}\psi + pz^{p-1}E(\psi) \tag{1.17}$$

where $E(\psi)$ denotes our error term and z is given by (1.12). It is well known that w and w' are the eigenvectors of the approximating linear operator

$$L_0\psi := -\frac{1}{pw^{p-1}}(\psi_{xx} - \psi + pw^{p-1}\psi)$$

corresponding to the eigenvalues $\lambda_{-1} < 0, \lambda_0 = 0$ of this operator, respectively. It is also well known that all the other eigenvalues of L_0 are positive.

(3) In the first part of the article we study the linear problem

$$pz^{p-1}\partial_t\psi = \psi_{xx} - \psi + pz^{p-1}\psi + pz^{p-1}f. \tag{1.18}$$

Assuming certain orthogonality conditions on f with respect to the eigenvectors w and w' of L_0, we establish the existence of an ancient solution to the linear problem (1.18), satisfying the appropriate energy and L^2 estimates. The latter means that we can bound the weighted L^2 norm of a solution in terms of the weighted L^2 norm of the right-hand side f. We also establish certain weighted $W^{2,\sigma}$ estimates for solutions to (1.18). It follows that the solution ψ belongs to the Banach space which is the intersection of these L^2 and weighted $W^{2,\sigma}$ spaces.

We denote by T the linear operator between our defined Banach spaces, so that $T(f)$ is the solution to the linear problem (1.18) satisfying the appropriate orthogonality conditions.

(4) In the second part of the article we study the nonlinear equation (1.17). We apply our linear theory to the nonlinear equation to establish the existence of a solution ψ to (1.17), by solving the equation $T(E(\psi)) = \psi$. We first show that we can achieve this, assuming that $E(\psi)$ satisfies our orthogonality assumptions with respect to w and w'. The main tool in this proof is the fixed point Theorem and subtle estimates of the error terms in our norms.

(5) In the final part of our proof we show how to adjust the parameters $\eta(t)$ and $h(t)$ so that the error term $E(\psi)$ in (1.17) indeed satisfies our orthogonality conditions. We see that this is equivalent to solving a certain nonlinear system of ODEs for $\eta(t), h(t)$. We establish the existence of solutions to this system by the fixed point Theorem and subtle estimates.

2. The ansatz of the problem

Following the discussion in the Introduction we look for an ancient solution u to equation (1.7). Since the long time existence for the Yamabe flow is well understood, it will be sufficient to construct a solution u which is defined on $\mathbb{R} \times (-\infty, t_0]$ with t_0 sufficiently close to $-\infty$. Hence, from now on we will restrict our attention to equation

$$(u^p)_t = u_{xx} - u + u^p, \quad (x,t) \in \mathbb{R} \times (-\infty, t_0], \tag{2.1}$$

with exponent $p = \frac{n+2}{n-2} > 1$.

2.1. The ansatz of our construction. We seek for a solution of (1.7) which is of the form

$$u(x,t) = (1 + \eta(t))z(x,t) + \psi(x,t)$$

for a suitable parameter function $\eta(t)$, where

$$z(x,t) = \sum_{j=1}^{2} w(x - \xi_j(t)) = \sum_{j=1}^{2} w_j$$

and $\psi(x,t) \to 0$ as $t \to -\infty$ in a certain sense. We recall that $w(x)$ is given by (1.9) and solves the equation (1.8). The functions $\xi_j(t)$ are given by

$$\xi_1(t) = -\xi(t) \quad \text{and} \quad \xi_2(t) = \xi(t)$$

where

$$\xi(t) = \xi_0(t) + h(t), \quad \xi_0(t) := \frac{1}{2}\log(2b|t|), \tag{2.2}$$

for a suitable parameter function $h(t)$ and a suitable constant $b > 0$. Both parameter functions $h(t)$ and $\eta(t)$ will decay in $|t|$, as $|t| \to \infty$ and will be chosen in Section 5.

Set

$$w_1 := w(x - \xi(t)), \quad w_2 := w(x + \xi(t)) \tag{2.3}$$

and

$$\bar{z}(x,t) = w'(x - \xi(t)) - w'(x + \xi(t)) = \partial_x w_1 - \partial_x w_2. \tag{2.4}$$

Also set

$$\tilde{z}(x,t) := (1 + \eta(t))z(x,t).$$

We notice that $z(\cdot,t)$ is an even function of x and we impose the condition that $\psi(\cdot,t)$ is an even function of x as well. Equation (1.7) then becomes

$$\partial_t(\tilde{z} + \psi)^p = (\partial_x^2\psi - \psi + \partial_x^2\tilde{z} - \tilde{z}) + (\tilde{z} + \psi)^p.$$

Using that $\partial_x^2 w_j - w_j = -w_j^p$, we obtain the equation

$$\partial_t(\tilde{z} + \psi)^p = \left(\partial_x^2\psi - \psi - (1 + \eta(t))\sum_{j=1}^{2} w_j^p\right) + (\tilde{z} + \psi)^p$$

which can be re-written as

$$pz^{p-1}\partial_t\psi = \partial_{xx}\psi - \psi + pz^{p-1}\psi - z^{p-1}C(\psi) + z^{p-1}E(\psi) \tag{2.5}$$

where $C(\psi)$ is a correction term that will be chosen in (3.10). The error term $E(\psi)$ is given by

$$E(\psi) := z^{1-p}M + \underbrace{C(\psi) + z^{1-p}\big[(1-\partial_t)N(\psi) - p\psi\partial_t z^{p-1}\big]}_{=:\,Q(\psi)} \tag{2.6}$$

where

$$M := \tilde{z}^p - \big[(1+\eta(t))[w^p(x+\xi(t)) + w^p(x-\xi(t))]\big] - \partial_t\tilde{z}^p \tag{2.7}$$

is the error term that is independent of ψ, and

$$N(\psi) := (\tilde{z}+\psi)^p - \tilde{z}^p - p\tilde{z}^{p-1}\psi + p\psi(\tilde{z}^{p-1} - z^{p-1}). \tag{2.8}$$

Our goal is to construct an ancient solution ψ of the above equation (2.5) with the aid of the linear theory for equation (2.5) that will be developed in Section 3. The solution ψ will be an even function in x and it will satisfy the orthogonality conditions

$$\int_{-\infty}^{\infty}\psi(x-\xi(t),t)w'(x)w^{p-1}\,dx = 0 \quad \text{for a.e. } t < t_0 \tag{2.9}$$

and

$$\int_{-\infty}^{\infty}\psi(x-\xi(t),t)w(x)w^{p-1}\,dx = 0 \quad \text{for a.e. } t < t_0. \tag{2.10}$$

The *correction term* $C(\psi)$ in equation (2.5) will be chosen in Section 3, in such a way so that the orthogonality conditions (2.9)–(2.10) for ψ are being preserved by equation (2.5) if the forcing term $E(\psi)$ satisfies the same conditions.

However, because in general the error term $E(\psi)$ may not satisfy the orthogonality conditions (2.9)–(2.10), we will first consider the auxiliary equation

$$\begin{aligned} pz^{p-1}\partial_t\psi = \partial_{xx}\psi - \psi + pz^{p-1}\psi - z^{p-1}C(\psi) \\ + z^{p-1}[E(\psi) - (c_1(t)z + c_2(t)\bar{z})] \end{aligned} \tag{2.11}$$

where $c_1(t)$ and $c_2(t)$ are chosen so that

$$\bar{E}(\psi) := E(\psi) - (c_1(t)z + c_2(t)\bar{z}) \tag{2.12}$$

satisfies the orthogonality conditions (2.9)–(2.10).

In Section 5 we will choose the parameter functions h and η so that $c_1(t) \equiv 0$ and $c_2(t) \equiv 0$. The parameter functions h and η will decay in t, as $t \to -\infty$, in certain norms that will be defined in Definition 2.7.

2.2. Norms. We will next introduce all the norms that will be used throughout the article. We will also fix the values of the various parameters. For a number $\tau < t_0 - 1$ we set $\Lambda_\tau = \mathbb{R} \times [\tau, \tau+1]$.

We first define the appropriate L^2, H^1 and H^2 norms.

Definition 2.1 (Local in time weighted L^2, H^1 and H^2 norms). Define

$$\begin{aligned}
\|\psi(\cdot,\tau)\|_{L^2} &= \left(\int_{-\infty}^{\infty} |\psi(\cdot,\tau)|^2 z^{p-1}\,dx\right)^{\frac{1}{2}},\\
\|\psi\|_{L^2(\Lambda_\tau)} &= \left(\iint_{\Lambda_\tau} |\psi|^2 z^{p-1}\,dx\,dt\right)^{\frac{1}{2}},\\
\|\psi\|_{H^1(\Lambda_\tau)} &= \|\psi\|_{L^2(\Lambda_\tau)} + \|z^{-\frac{p-1}{2}}\psi_x\|_{L^2(\Lambda_\tau)},\\
\|\psi\|_{H^2(\Lambda_\tau)} &= \|\psi_t\|_{L^2(\Lambda_\tau)} + \|z^{-(p-1)}(\psi_{xx}-\psi)\|_{L^2(\Lambda_\tau)}\\
&\quad + \|z^{-\frac{p-1}{2}}\psi_{xx}\|_{L^2(\Lambda_\tau)} + \|\psi\|_{H^1(\Lambda_\tau)}.
\end{aligned}$$

Definition 2.2 (Global in time weighted L^2, H^1 and H^2 norms). For a given number $\nu \in [0,1)$ we define

$$\begin{aligned}
\|\psi\|^{\nu}_{L^2_{t_0}} &= \sup_{\tau\le t_0-1} |\tau|^{\nu}\|\psi\|_{L^2(\Lambda_\tau)},\\
\|\psi\|^{\nu}_{H^1_{t_0}} &= \sup_{\tau\le t_0-1} |\tau|^{\nu}\|\psi\|_{H^1(\Lambda_\tau)},\\
\|\psi\|^{\nu}_{H^2_{t_0}} &= \sup_{\tau\le t_0-1} |\tau|^{\nu}\|\psi\|_{H^2(\Lambda_\tau)}.
\end{aligned}$$

Also, for any $s < t_0 - 1$, we define

$$\begin{aligned}
\|\psi\|^{\nu}_{L^2_{s,t_0}} &= \sup_{s\le\tau\le t_0-1} |\tau|^{\nu}\|\psi\|_{L^2(\Lambda_\tau)},\\
\|\psi\|^{\nu}_{H^1_{s,t_0}} &= \sup_{s\le\tau\le t_0-1} |\tau|^{\nu}\|\psi\|_{H^1(\Lambda_\tau)},\\
\|\psi\|^{\nu}_{H^2_{s,t_0}} &= \sup_{s\le\tau\le t_0-1} |\tau|^{\nu}\|\psi\|_{H^2(\Lambda_\tau)}.
\end{aligned}$$

When $\nu = 0$, we will omit the superscript ν.

Set

$$\beta := \frac{2}{n-2} = \frac{p-1}{2}.$$

For a given number $\sigma \ge 2$, we next give the definition of the weighted $W^{2,\sigma}$ norm and σ will be chosen later in the text. To this end, we define the weight function $\alpha_\sigma(x,t)$ by

$$\alpha_\sigma(x,t) = \begin{cases} z^{n\beta-\sigma}(x,t) & \text{if } |x| > \xi(t),\\ z^{(2\beta+\theta)\sigma}(x,t) & \text{if } |x| \le \xi(t), \end{cases} \tag{2.13}$$

where θ is a small positive number which will be chosen sufficiently close to zero.

Remark 2.1. (a) We will see in the sequel that the weight function in the outer region $|x| > \xi(t)$ is such that the solution ϕ of (1.18), or equivalently the solution u of the nonlinear problem, corresponds to a smooth solution, when lifted up to the sphere. However, it is necessary to change the weight function α_σ in the inner region $|x| \le \xi(t)$ to incorporate the singularity, as $t \to -\infty$, of the solution u of our nonlinear problem in that region.

(b) In the transition region $x = \pm\xi(t) + O(1)$ the two weights are equivalent.

Definition 2.3 (Local in time weighted $W^{2,\sigma}$ norms). For $\sigma \geq 2$ define

$$\|\psi\|_{\sigma,\Lambda_\tau} = \left(\iint_{\Lambda_\tau} |\psi|^\sigma \alpha_\sigma \, dx \, dt \right)^{\frac{1}{\sigma}}$$

and

$$\|\psi\|_{2,\sigma,\Lambda_\tau} = \|\psi_t\|_{\sigma,\Lambda_\tau} + \|\psi\|_{\sigma,\Lambda_\tau} + \|\psi_x\|_{\sigma,\Lambda_\tau} + \|\psi_{xx}\|_{\sigma,\Lambda_\tau}.$$

Definition 2.4 (Global in time weighted $W^{2,\sigma}$ norms). For a fixed number $\nu \in [0,1)$, $\sigma \geq 2$ and Λ_τ as above, we define

$$\|\psi\|^\nu_{\sigma,t_0} = \sup_{\tau \leq t_0 - 1} |\tau|^\nu \|\psi\|_{\sigma,\Lambda_\tau}$$

and

$$\|\psi\|^\nu_{2,\sigma,t_0} = \sup_{\tau \leq t_0 - 1} |\tau|^\nu \|\psi\|_{2,\sigma,\Lambda_\tau}.$$

We will next define a weighted L^∞ norm and our global norm.

Definition 2.5 (Weighted L^∞ norm). For a given $\nu \in [0,1]$, we define the norm

$$\|\psi\|^\nu_{L^\infty_{t_0}} = \sup_{t \leq t_0} |\tau|^\nu \|\psi(\cdot,\tau)\|_{L^\infty(\mathbb{R})}$$

and the weighted L^∞ norm as

$$\|\psi\|^\nu_{\infty,t_0} = \|z^{-1}\psi\chi_{\{|x|>\xi(t)\}}\|^\nu_{L^\infty_{t_0}} + \|\psi\chi_{\{|x|\leq\xi(t)\}}\|^\nu_{L^\infty_{t_0}}.$$

We finally define the global norm for the perturbation term ψ.

Definition 2.6 (Global norm). For $\nu \in [0,1)$ and $\sigma \geq 2$ we define the norms

$$\|\psi\|^\nu_{*,\sigma,t_0} = \|\psi\|^\nu_{L^2_{t_0}} + \|\psi\|^\nu_{\sigma,t_0}$$

and

$$\|\psi\|^\nu_{*,2,\sigma,t_0} = \|\psi\|^\nu_{H^2_{t_0}} + \|\psi\|^\nu_{2,\sigma,t_0} + \|\psi\|^\nu_{\infty,t_0}.$$

Also, for any $\tau < t_0$, we denote by

$$\|\psi\|_{*,\sigma,\Lambda_\tau} = \|\psi\|_{L^2(\Lambda_\tau)} + \|\psi\|_{\sigma,\Lambda_\tau}.$$

We will next define the norms for the parameters $\eta(t)$ and $h(t)$. They are more or less determined by the choice of the global norm for ψ.

Definition 2.7 (Weighted in time norms). For $\mu \in [0,1)$ and $\sigma \geq 2$, and for any functions η and h defined on $(-\infty, t_0]$, we define the norms

$$\begin{aligned}
\|\eta\|^\mu_{\sigma,t_0} &= \sup_{\tau \leq t_0 - 1} |\tau|^\mu \left(\int_\tau^{\tau+1} |\eta(t)|^\sigma \, dt \right)^{\frac{1}{\sigma}}, \\
\|\eta\|^\mu_{\infty,t_0} &= \sup_{\tau \leq t_0} (|\tau|^\mu |\eta(\tau)|), \\
\|\eta\|^\mu_{1,\sigma,t_0} &= \|\eta\|^\mu_{\infty,t_0} + \|\dot\eta\|^\mu_{\sigma,t_0}, \\
\|h\|^{\mu,1+\mu}_{1,\sigma,t_0} &= \|h\|^\mu_{\infty,t_0} + \|\dot h\|^{1+\mu}_{\sigma,t_0}.
\end{aligned}$$

2.3. Outline of our construction. We will conclude this section by outlining the construction of the solution u.

Definition 2.8. We define X to be the Banach space of all functions ψ on $\mathbb{R} \times (-\infty, t_0]$ with $\|\psi\|_{*,2,\sigma,\nu} < \infty$ which also satisfy the orthogonality conditions (2.9)–(2.10).

We denote by T the linear operator which assigns to any given f with $\|f\|^{\nu}_{*,\sigma,t_0} < \infty$ the solution $\psi := T(f)$ of the linear auxiliary equation

$$pz^{p-1}\partial_t \psi = \partial_{xx}\psi - \psi + pz^{p-1}\psi - z^{p-1}C(\psi) + z^{p-1} f,$$

with the orthogonality conditions (2.9)–(2.10) being satisfied by f and ψ and with $C(\psi)$ given by (3.10). The construction of such a function ψ will be given in Section 3.

Going back to the nonlinear problem, a function ψ is a solution of (2.11) if and only if $\psi \in X$ solves the fixed point problem

$$\psi = A(\psi) \tag{2.14}$$

where

$$A(\psi) := T(\bar{E}(\psi))$$

and $\bar{E}(\psi)$ is as in (2.12).

Outline. Given any parameter functions (h, η) with $\|h\|^{\mu,1+\mu}_{1,\sigma,t_0} < \infty$ and $\|\eta\|^{\mu}_{\sigma,t_0} < \infty$, we will establish, in Section 4, the existence of a solution $\psi := \Psi(h, \eta)$ of the fixed point problem (2.14). In the last Section 5 we will choose the parameter functions h and η so that $c_1(t) \equiv 0$ and $c_2(t) \equiv 0$. We will conclude that the solution ψ of (2.14) which is equivalent to (2.11) is actually a solution to (2.5). Hence, $u := (1+\eta)z + \psi$ will be the desired ancient solution to (1.7).

2.4. Notation. We summarize now the notation of parameters, functions and norms used throughout the article.

Notation 2.1. The choice of the parameters $p, \beta, \sigma, \nu, \mu, b$ and θ:

(i) For given dimension $n \geq 3$, we recall that

$$p := \frac{n+2}{n-2} \quad \text{and} \quad \beta := \frac{2}{n-2} = \frac{p-1}{2}.$$

(ii) In Theorem 1.1, $\sigma = n+2$. We choose ν so that $\frac{1}{2} < \nu < \min\{\nu_0, 1\}$, where $\nu_0 = \nu_0(n)$ is determined by the estimate of Lemma 4.1. We choose $0 < \mu < \min\{2\nu - 1, \gamma\}$, where $\gamma = \gamma(n) \in (0,1)$ is determined by Lemma 5.1. The constant $b = b(n) > 0$ is defined in (5.3).

(iii) The constant θ in (2.13) is a small positive constant as determined in the proof of Proposition 3.3. The above constants are all universal depending only on the dimension n.

Notation 2.2. The choice of functions:

(i) We denote by $w(x)$ the solution to (1.8) given by (1.9).

(ii) The function $\xi_0(t)$ and the function $\xi(t)$ (for a parameter function $h(t)$) are given in (7.2).

(iii) The functions $z(x,t)$ and $w_1(x,t)$, $w_2(x,t)$ are defined in (1.12) and (2.3) respectively.

(iv) Throughout the article, $u(x,t)$ will denote an ancient solution of the nonlinear equation (1.7) of the form (1.11) defined on $\mathbb{R}^n \times (-\infty, t_0]$, where t_0 is a constant which will be chosen sufficiently close to $-\infty$. $\eta(t)$ is a parameter function defined on $(-\infty, t_0]$. The perturbation function $\psi(x,t)$ satisfies equation (2.5) where $E(\psi)$ is a nonlinear error term give by (2.6)-(2.8).

(v) Only in Section 3, $\psi(x,t)$ will denote a solution to equation (3.9), for a given f, where the correction term $C(\psi)$ is given by (3.10). Also, $\psi^s(x,t)$ will denote a solution of equation (3.11).

Notation 2.3. The norms:

(i) For a given $\tau < t_0$, the norm $\|\psi(\cdot,\tau)\|_{L^2}$ is given in Definition 2.1.

(ii) For any $\tau < t_0 - 1$ we set $\Lambda_\tau := \mathbb{R} \times [\tau, \tau+1]$. For a given function $\psi(x,t)$ defined on Λ_τ, the norms $\|\psi\|_{L^2(\Lambda_\tau)}$, $\|\psi\|_{H^1(\Lambda_\tau)}$ and $\|\psi\|_{H^1(\Lambda_\tau)}$ are given in Definition 2.1.

(iii) The norms $\|\psi\|^\nu_{L^2_{t_0}}$, $\|\psi\|^\nu_{H^1_{t_0}}$ and $\|\psi\|^\nu_{H^2_{t_0}}$ are given in Definition 2.2.

(iv) The norms $\|\psi\|_{\sigma,\Lambda_\tau}$, $\|\psi\|_{2,\sigma,\Lambda_\tau}$ are given in Definition 2.3, while the norms $\|\psi\|^\nu_{\sigma,t_0}$, $\|\psi\|^\nu_{2,\sigma,t_0}$ are given in Definition 2.4.

(v) The weighted L^∞ norm $\|\psi\|^\nu_{L^\infty_{t_0}}$ is given in Definition 2.5.

(vi) The global norms $\|f\|^\nu_{*,2,\sigma,t_0}$, $\|w\|^\nu_{*,\sigma,t_0}$ are given in Definition 2.6.

(vii) For given functions $h(t)$ and $\eta(t)$, the norms $\|h\|^{\mu,1+\mu}_{1,\sigma,t_0}$ and $\|\eta\|^\mu_{\sigma,t_0}$ are given in Definition 2.7.

3. The linear equation

Consider the linear equation

$$pz^{p-1}\partial_t\psi = \partial_{xx}\psi - \psi + pz^{p-1}\psi + z^{p-1}g \tag{3.1}$$

defined on $-\infty < t \le t_0$. The coefficient z is given by

$$z(x,t) = w(x-\xi(t)) + w(x+\xi(t)) \tag{3.2}$$

where $\xi(t)$ is given by (7.2) for a suitable function $h \in C^1((-\infty, t_0])$ and $b > 0$. Note that $z(\cdot,t)$ is even in x. We will also impose that $g(\cdot,t)$ is an even function in x and we shall seek for a solution $\psi(\cdot,t)$ which is even in x. We will consider a class of functions g defined for $(x,t) \in \mathbb{R} \times (-\infty, t_0]$ that decay both in x and t at suitable rates and satisfy certain orthogonality conditions, and will build a solution ψ that defines a linear operator of g which shares the same decay rates.

Our goal is to establish the existence of the solution ψ of (3.1) in appropriate L^2 and H^1 spaces, defined in Definition 2.2. We observe that in the region $-\infty < x < -\xi(t)$ and under the change of variables $\bar{x} := x + \xi(t)$ the operator in (3.1), namely

$$L\psi := \frac{1}{z^{p-1}}(\psi_{xx} - \psi + pz^{p-1}\psi)$$

can be approximated by the elliptic operator

$$L_0\phi := \frac{1}{w^{p-1}}(\phi_{\bar{x}\bar{x}} - \phi + pw^{p-1}\phi), \tag{3.3}$$

with $\phi(\bar{x}, t) := \psi(x, t)$, since $z(x, t) = w(\bar{x}) + w(\bar{x} - 2\xi(t)) \approx w(\bar{x})$ in that region. Defining $\bar{g}(\bar{x}, t) := g(x, t)$, the approximated parabolic equation takes the form

$$pw^{p-1}\partial_t\phi = \partial_{\bar{x}\bar{x}}\phi - \phi_{\bar{x}}\dot{\xi} - \phi + pw^{p-1}\phi + w^{p-1}\bar{g}. \tag{3.4}$$

The region $\xi(t) < x < +\infty$, under the change of variables $\bar{x} := x - \xi(t)$ is treated similarly.

We wish to construct an ancient solution ψ of (3.1) such that the weighted L^2 norm of ψ is controlled by the weighted L^2 norm of the right-hand side g. First, let us consider the eigenvalue problem

$$L_0\theta + \lambda\theta = 0, \quad \theta \in S,$$

on the weighted space $L^2(w^{p-1}dx)$. It is standard that this problem has an infinite sequence of simple eigenvalues

$$\lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$$

with an associated orthonormal basis of the space $L^2(w^{p-1}dx)$ constituted by eigenfunctions θ_j, $j = -1, 0, 1, \ldots$, where θ_{-1} is a suitable multiple of w and θ_0 of w'. Since we are seeking for a solution which is controlled by the weighted L^2 norm of its right-hand side, we need to restrict ourselves to a subspace $S_0 \subset L^2(w^{p-1}dx)$ which constitutes of functions $g(\cdot, t)$ on $\mathbb{R} \times (-\infty, t_0]$ that are even in x and that also satisfy the orthogonality conditions

$$\int_{-\infty}^{\infty} g(\bar{x} - \xi(t), t)w'(\bar{x})w^{p-1}\, d\bar{x} = 0 \quad \text{for a.e. } t < t_0 \tag{3.5}$$

and

$$\int_{-\infty}^{\infty} g(\bar{x} - \xi(t), t)w(\bar{x})w^{p-1}\, d\bar{x} = 0 \quad \text{for a.e. } t < t_0. \tag{3.6}$$

Notice that since g is an even function in x, then the orthogonality conditions (3.5) and (3.6) also imply the symmetric conditions

$$\int_{-\infty}^{\infty} g(\bar{x} + \xi(t), t)w'(\bar{x})w^{p-1}\, d\bar{x} = 0 \quad \text{for a.e. } t < t_0 \tag{3.7}$$

and

$$\int_{-\infty}^{\infty} g(\bar{x} + \xi(t), t)w(\bar{x})w^{p-1}\, d\bar{x} = 0 \quad \text{for a.e. } t < t_0. \tag{3.8}$$

This easily follows by changing the variables of integration and using that w is an even function of $\bar{x}$.

We wish to establish the existence of an ancient solution of (3.1) on $\mathbb{R} \times (-\infty, t_0]$ which satisfies the estimate

$$\sup_{\tau \le t_0} |\tau|^{\nu} \|\psi(\tau)\|_{L^2} \le C\|g\|^{\nu}_{L^2_{t_0}}.$$

Such a solution can be easily constructed for the approximated equation (3.4) if $\bar{g} \in S_0$. Indeed, one simply looks for a solution in the form

$$\phi(\bar{x}, t) = \sum_{j=1}^{\infty} \phi_j(t)\theta_j(\bar{x})$$

where θ_j, $j \geq 1$, are the eigenfunctions, corresponding to the positive eigenvalues λ_j, $j \geq 1$ mentioned above. However, this cannot be done for equation (3.1) as its coefficients depend on time and as a result the equation does not preserve the orthogonality conditions (3.5) and (3.6). In order to achieve our goal we need to consider the equation

$$pz^{p-1}\partial_t \psi = \partial_{xx}\psi - \psi + pz^{p-1}\psi + z^{p-1}[f - C(\psi)] \tag{3.9}$$

where $g := f - C(\psi)$ where $C(\psi)$ has the form

$$C(\psi) = d_1(t)z(x,t) + d_2(t)\bar{z}(x,t). \tag{3.10}$$

Recall that $z(x,t) := w(x - \xi(t)) + w(x + \xi(t))$ and that $\bar{z}$ is defined by (2.4).

We will construct an ancient solution of (3.9) on $\mathbb{R} \times (-\infty, t_0]$, by considering first the solution ψ^s of the initial value problem

$$\begin{cases} pz^{p-1}\partial_t \psi^s = \psi^s_{xx} - \psi^s + pz^{p-1}\psi^s + z^{p-1}(f - C(\psi^s,t)) & \text{on } \mathbb{R}\times[s,t_0], \\ \psi^s(\cdot, s) = 0 & \text{on } \mathbb{R}, \end{cases} \tag{3.11}$$

and then pass to the limit as $s \to -\infty$. The existence of ψ^s will be shown in Lemma 3.1.

The coefficients $d_1(t), d_2(t)$ in (3.9) are defined so that $\psi^s(\cdot, t) \in S_0$ for all $t \in [s, t_0]$. We will next determine the coefficients d_1 and d_2. To this end, it is more convenient to work with the function

$$\phi^s(x,t) := \psi^s(x - \xi(t), t).$$

To simplify notation we omit for the moment the superscript s and set $\phi = \phi^s$ and $\psi = \psi^s$. A direct computation shows that if ψ is a solution to (3.9), then the function ϕ satisfies the equation

$$p\partial_t \phi = L_0\phi + E(\phi) + \bar{f} - d_1\bar{w} - d_2\tilde{w}. \tag{3.12}$$

Here we have used the following notation:

$$\bar{w}(x,t) := z(x - \xi(t), t) = w(x) + w(x - 2\xi(t))$$

and

$$\tilde{w} := \bar{z}(x - \xi(t)) = w'(x) - w'(x - 2\xi(t))$$

and $\bar{f}(x,t) := f(x - \xi(t), t)$, while $E(\phi)$ denotes the error term

$$E(\phi) := -\dot{\xi}(t)\phi_x + \left(\frac{1}{\bar{w}^{p-1}} - \frac{1}{w^{p-1}}\right)(\phi_{xx} - \phi).$$

We recall that L_0 is given by (3.3). Also recall that θ_i, $i = -1, 0$, denote the eigenfunctions (which are the multiples of w, w') of operator L_0, corresponding to the eigenvalues $\lambda_{-1} < 0$ and $\lambda_0 = 0$, respectively. We have assumed that $\bar{f}$ is orthogonal to θ_i, $i = -1, 0$, namely

$$\int_{-\infty}^{\infty} \bar{f}(x)\theta_i(x)w^{p-1}\,dx = 0.$$

Since $\phi(\cdot, s) = 0$ (remember that $\phi = \phi^s$ for the moment), it follows from (3.12) that the solution ϕ will remain orthogonal to the eigenfunctions θ_i, $i = -1, 0$, if and only if the coefficients $d_1(t)$ and $d_2(t)$ satisfy the system of equations

$$d_1(t)a_1^i(t) + d_2(t)a_2^i(t) = E^i, \quad i = -1, 0, \tag{3.13}$$

where

$$\begin{aligned} a_1^i(t) &= \int_{-\infty}^{\infty} \bar{w}(x,t)\theta_i(x) w^{p-1}\, dx, \\ a_2^i(t) &= \int_{-\infty}^{\infty} \tilde{w}(x,t)\theta_i(x) w^{p-1}\, dx, \\ E^i &= \int_{-\infty}^{\infty} E(\phi)(x,t)\theta_i(x) w^{p-1}\, dx, \quad i = -1, 0. \end{aligned}$$

Using that $\bar{w}(x,t) := w(x) + w(x - 2\xi(t))$ and $\tilde{w}(x,t) := w'(x) - w'(x - 2\xi(t))$ together with the orthogonality

$$\int_{-\infty}^{\infty} w(x)\theta_0(x) w^{p-1}\, dx = \int_{-\infty}^{\infty} w'(x)\theta_{-1}(x) w^{p-1}\, dx = 0,$$

we conclude that

$$\begin{aligned} a_1^i(t) &= e_1^i + \int_{-\infty}^{\infty} w(x - 2\xi(t))\theta_i(x) w^{p-1}\, dx, \\ a_2^i(t) &= e_2^i - \int_{-\infty}^{\infty} w'(x - 2\xi(t))\theta_i(x) w^{p-1}\, dx \end{aligned}$$

where

$$e_1^{-1} = c_{-1}\int_{-\infty}^{\infty} w^{p+1}\, dx > 0, \quad e_2^0 = c_0 \int_{-\infty}^{\infty} w'(x)^2 w^{p-1}\, dx > 0, \quad e_1^0 = e_2^{-1} = 0.$$

It is easy to see that

$$\begin{aligned} \int_{-\infty}^{\infty} w(x - 2\xi(t))\theta_i(x) w^{p-1}\, dx &= O(|t|^{-1}), \\ \int_{-\infty}^{\infty} w'(x - 2\xi(t))\theta_i(x) w^{p-1}\, dx &= O(|t|^{-1}), \end{aligned}$$

as $t \to -\infty$. Hence,

$$\begin{aligned} a_1^{-1}(t) &= e_1^{-1} + O(|t|^{-1}), \\ a_2^0(t) &= e_2^0 + O(|t|^{-1}), \\ a_1^0 &= O(|t|^{-1}), \\ a_2^{-1} &= O(|t|^{-1}). \end{aligned}$$

It follows that the determinant D of the coefficients of system (3.13) satisfies

$$D := a_1^{-1} a_2^0 - a_1^0 a_2^{-1} = e_1^{-1} e_2^0 + O(|t|^{-1}) > 0, \quad \text{as } t \to -\infty.$$

Solving system (3.13) gives

$$d_1(t) = \frac{a_2^0(t) E^{-1}(t) - a_2^{-1}(t) E^0(t)}{D} = \frac{e_0^2 E^{-1}(t)}{D} + O(|t|^{-1})$$

and

$$d_2(t) = \frac{a_1^{-1}(t) E^{-1}(t) - a_1^0(t) E^0(t)}{D} = \frac{e_1^{-1} E^0(t)}{D} + O(|t|^{-1}).$$

Claim 3.1. *We have*

$$\left(\int_\tau^{\tau+1}(d_1^2+d_2^2)\,dt\right)^{\frac{1}{2}} \leq C\frac{1}{\sqrt{|\tau|}}\left\{\left(\iint_{\Lambda_\tau}\left(\frac{\phi_{xx}-\phi}{\bar{w}^{p-1}}\right)^2\bar{w}^{p-1}\,dx\,dt\right)^{\frac{1}{2}}+\|\phi\|_{L^\infty(\Lambda_\tau)}\right\}. \tag{3.14}$$

Proof. It is easy to see that

$$\left(\int_\tau^{\tau+1}(d_1^2+d_2^2)\,dt\right)^{\frac{1}{2}} \leq C\left(\int_\tau^{\tau+1}((E^{-1})^2+(E^0)^2)\,dt\right)^{\frac{1}{2}}+O(|\tau|^{-1}) \tag{3.15}$$

so it is enough to estimate the integrals $\int_\tau^{\tau+1}(E^i)^2\,dt$, for $i=\{-1,0\}$ and $\tau\leq t_0$. We will only discuss the computation for E^0, as the computation for E^{-1} is identical. We have

$$E^0=-\dot{\xi}(t)\int_{-\infty}^{\infty}\phi_x w'(x)w^{p-1}\,dx+\int_{-\infty}^{\infty}c(x,t)(\phi_{xx}-\phi)w'(x)\,dx=E_1^0+E_2^0$$

with

$$c(x,t)=\frac{w^{p-1}-\bar{w}^{p-1}}{\bar{w}^{p-1}}=\left(\frac{w}{\bar{w}}\right)^{p-1}-1.$$

Clearly, we have

$$\begin{aligned}\left(\int_\tau^{\tau+1}|E_1^0|^2\,dt\right)^{\frac{1}{2}} &\leq C\left(\int_\tau^{\tau+1}|\dot{\xi}|^2\left(\int_{-\infty}^{\infty}\phi_x w'(x)w^{p-1}\,dx\right)^2dt\right)^{\frac{1}{2}}\\ &= C\left(\int_\tau^{\tau+1}|\dot{\xi}|^2\left(\int_{-\infty}^{\infty}\phi(w'(x)w^{p-1})_x\,dx\right)^2dt\right)^{\frac{1}{2}}\\ &\leq C\|\phi\|_{L^\infty(\Lambda_\tau)}\left(\int_\tau^{\tau+1}|\dot{\xi}|^2\,dt\right)^{\frac{1}{2}}\\ &\leq \frac{C}{|\tau|}\|\phi\|_{L^\infty(\Lambda_\tau)}.\end{aligned} \tag{3.16}$$

For the second term, we have

$$\int_{-\infty}^{\infty}c(x,t)(\phi_{xx}-\phi)w'(x)\,dx\leq I(t)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\frac{(\phi_{xx}-\phi)^2}{\bar{w}^{p-1}}\,dx\right)^{\frac{1}{2}} \tag{3.17}$$

where

$$I(t):=\int_{-\infty}^{\infty}c^2(x,t)|w'(x)|^2\bar{w}^{p-1}\,dx.$$

Recall that ξ given by (7.2) satisfies $\xi(t)=\frac{1}{2}\log|t|+O(1)$, as $t\to-\infty$. On $x<\xi(t)$ we have $w\leq\bar{w}\leq 2w$, hence

$$\frac{1}{2}\leq\frac{w}{\bar{w}}\leq 1.$$

It follows that

$$c^2(x,t)=\left(1-\left(\frac{w}{\bar{w}}\right)^{p-1}\right)^2\leq C(p)\left(1-\frac{w}{\bar{w}}\right)^2.$$

We conclude that

$$\begin{aligned}
I_1 &:= \int_{-\infty}^{\xi(t)} c^2(x,t)|w'(x)|^2 \bar{w}^{p-1}\,dx \\
&\le C \int_{-\infty}^{\xi(t)} \Big(\frac{w(x-2\xi)}{\bar{w}(x,t)}\Big)^2 |w'(x)|^2 \bar{w}^{p-1}\,dx \\
&\le C \int_{-\infty}^{\xi(t)} w(x-2\xi)^2 \bar{w}^{p-1}\,dx \\
&\le C|t|^{-2}\Big(\int_{-\infty}^{0} e^{2x}e^{(p-1)x}\,dx + \int_0^{\xi(t)} e^{2x}e^{-(p-1)x}\,dx\Big) \\
&\le C|t|^{-2}\big(C_1 + C_2|t|^{\frac{3-p}{2}}\big) \\
&\le C\max\big\{|t|^{-2}, |t|^{-\frac{1+p}{2}}\big\}.
\end{aligned}$$

On $x > \xi(t)$, using the bound $c^2 \le 1$ and $|w'(x)|^2 \le C|t|^{-1}$, we have

$$\begin{aligned}
I_2 &:= \int_{\xi(t)}^{\infty} c^2(x,t)|w'(x)|^2 \bar{w}^{p-1}\,dx \\
&\le C \int_{\xi(t)}^{\infty} |w'(x)|^2 \bar{w}^{p-1}\,dx \\
&\le C|t|^{-1} \int_{\xi(t)}^{\infty} \bar{w}^{p-1}\,dx \\
&\le C|t|^{-1}.
\end{aligned}$$

Since $p > 1$, combining the above gives us the estimate

$$I(t) = I_1 + I_2 \le C|t|^{-1}.$$

Using the last estimate in (3.17) yields the bound

$$|E_2^0(t)| \le C\frac{1}{\sqrt{|t|}}\Big(\int_{-\infty}^{\infty} \frac{(\phi_{xx}-\phi)^2}{\bar{w}^{p-1}}\,dx\Big)^{\frac{1}{2}}. \tag{3.18}$$

Combining (3.16) and (3.18) gives us the bound

$$\Big(\int_{\tau}^{\tau+1} |E^0(t)|^2\,dt\Big)^{\frac{1}{2}} \le C\frac{1}{\sqrt{|\tau|}}\Big\{\Big(\iint_{\Lambda_\tau} (\frac{\phi_{xx}-\phi}{\bar{w}^{p-1}})^2 \bar{w}^{p-1}\,dx\,dt\Big)^{\frac{1}{2}} + \|\phi\|_{L^\infty(\Lambda_\tau)}\Big\}$$

and the same bound holds for $E^{-1}(t)$. By (3.15) it follows that (3.14) holds. □

Using (3.14) we can easily estimate the $L^2(\Lambda_\tau)$ norm of the term

$$C(\psi) := d_1(t)z + d_2(t)\bar{z}$$

by the $H^2(\Lambda_\tau)$ norm of function ψ, as

$$\|C(\psi)\|_{L^2(\Lambda_\tau)} \le C\frac{1}{\sqrt{|\tau|}}(\|\psi\|_{H^2(\Lambda_\tau)} + \|\psi\|_{L^\infty(\Lambda_\tau)}). \tag{3.19}$$

The main result in this section is the following proposition.

Proposition 3.1. *For given numbers $p > 1$, $b > 0$ and $0 \le \mu \le 1$ and a given function $\xi := \frac{1}{2}\log(2b|t|) + h$ on $(-\infty, t_0]$ with $\|h\|^{\mu,1+\mu}_{1,\sigma,t_0} \le 1$, consider equation* (3.9) *with coefficient z given by* (3.2). *Then, for any $\nu \in [0,1]$, there is a number $t_0 < 0$, depending on μ, ν, b and p and such that for any even function f on $\mathbb{R} \times (-\infty, t_0]$ with*

$$\|f\|^{\nu}_{L^2_{t_0}} < \infty$$

satisfying the orthogonality conditions (3.5)–(3.6) *there exists an ancient solution ψ of* (3.9) *on $-\infty \le t \le t_0$ also satisfying the orthogonality conditions* (3.5)–(3.6)*, and for which*

$$\sup_{\tau \le t_0} |\tau|^{\nu} \|\psi(\tau)\|_{L^2} + \|\psi\|^{\nu}_{H^2_{t_0}} \le C \|f\|^{\nu}_{L^2_{t_0}}. \tag{3.20}$$

The constant C depends only on b, μ, ν and p.

As we already discussed, the ancient solution ψ will be constructed as the limit of the solutions ψ^s to (3.11), as $s \to -\infty$. The existence of the solutions ψ^s is given by the next lemma.

Lemma 3.1. *Under the assumptions of Proposition* 3.1*, there exists a number $t_0 < 0$ depending on b, μ, ν and p and a solution ψ^s of the initial value problem* (3.11) *also satisfying the orthogonality conditions* (3.5) *and* (3.6)*. In addition,*

$$\sup_{\tau \in [s,t_0]} |\tau|^{\nu} \|\psi^s(\tau)\|_{L^2} + \|\psi^s\|^{\nu}_{H^2_{s,t_0}} \le C \|f\|^{\nu}_{L^2_{s,t_0}} \tag{3.21}$$

where C depends only on b, ν, μ and p.

Remark 3.1 (Dependence on function ξ). For the remaining of Section 3, we will fix $b > 0$, $\mu \in (0,1)$ and a function $\xi := \frac{1}{2}\log(2b|t|) + h$ with $\|h\|^{\mu,1+\mu}_{1,\sigma,t_0} \le 1$ and we will only discuss the dependence of the various constants on s and t_0, while assuming that may also depend on b, μ and ν.

3.1. A priori estimates. We will establish in this subsection a priori L^2 and H^2 energy estimates for the solutions ψ^s of (3.11) that are independent on s. We begin by proving an energy estimate (independent of s) for solutions of the initial value problem

$$\begin{cases} p z^{p-1} \partial_t \psi^s = \psi^s_{xx} - \psi^s + p z^{p-1} \psi^s + z^{p-1} g & \text{on } \mathbb{R} \times [s, t_0], \\ \quad \psi^s(\cdot, s) = 0 & \text{on } \mathbb{R}. \end{cases} \tag{3.22}$$

Energy estimates for solutions of equation (3.11) will easily follow by Lemma 3.2 and estimate (3.19).

Lemma 3.2 (Energy H^2 and L^∞ estimate for equation (3.1)). *Let $\psi^s(x,t)$ be a solution of* (3.22)*. Then, for any $\nu \in [0,1)$ there exists a uniform in s constant C so that for $|t_0|$ sufficiently large we have*

$$\sup_{\tau \in [s,t_0]} |\tau|^{\nu} \|\psi\|_{L^\infty(\Lambda_\tau)} + \|\psi\|^{\nu}_{H^2_{s,t_0}} \le C \big(\|\psi\|^{\nu}_{L^2_{s,t_0}} + \|g\|^{\nu}_{L^2_{s,t_0}} \big). \tag{3.23}$$

Proof. To simplify the notation, we will denote ψ^s by ψ. In what follows we will perform various integration by parts in space without worrying about the boundary terms at infinity. This can be easily justified by considering approximating solutions ψ_R^s of the Dirichlet problem on expanding cylinders $Q_R := [-R, R] \times [s, t_0]$, establish the a priori estimates on ψ_R^s, independent of both, s and R, and then pass to the limit of ψ_R^s as $R \to \infty$ (our solution ψ^s in Lemma 3.1 will be constructed that way).

If we multiply equation (3.1) by ψ and integrate it over $\mathbb{R}$, we obtain,

$$\begin{aligned} \frac{p}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \psi^2 z^{p-1}\,dx \\ = \int_{-\infty}^{\infty}\Big(\psi\psi_{xx} - \psi^2 + p\Big[1 + (p-1)\frac{z_t}{z}\Big]\psi^2 z^{p-1} + g\psi z^{p-1}\Big)\,dx. \end{aligned} \tag{3.24}$$

If we integrate by parts the first term on the right-hand side, use the bound $\frac{|z_t|}{z} \le C|\dot\xi|$ and apply Cauchy–Schwarz, we obtain

$$\begin{aligned} \frac{p}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \psi^2(\cdot,t) z^{p-1}\,dx + \int_{-\infty}^{\infty}(\psi_x^2 + \psi^2)\,dx \\ \le C\Big(\int_{-\infty}^{\infty}(\psi^2 + g^2)z^{p-1}\,dx + |\dot\xi|\int_{-\infty}^{\infty}\psi^2 z^{p-1}\,dx\Big). \end{aligned}$$

For any number $\tau \in [s, t_0 - 1]$, set $\eta(t) = t - \tau$ so that $0 \le \eta(t) \le 1$ on $[\tau, \tau+1]$. Then, for any $t \in [\tau, \tau+1]$, we have

$$\begin{aligned} \frac{d}{dt}\Big(\eta(t)\int_{-\infty}^{\infty}\psi^2(\cdot,t)z^{p-1}\,dx\Big) + \eta(t)\int_{-\infty}^{\infty}(\psi_x^2 + \psi^2)\,dx \\ \le C\Big(\int_{-\infty}^{\infty}(\psi^2 + g^2)z^{p-1}\,dx + |\dot\xi|\int_{-\infty}^{\infty}\psi^2 z^{p-1}\,dx\Big). \end{aligned}$$

Integrating this inequality in time on $[\tau, \tau+1]$ while applying the Cauchy–Schwarz inequality to the last term and using that $\eta(t) \le 1$ and

$$\Big(\int_{\tau-1}^{\tau}|\dot\xi|^2\,dt\Big)^{\frac12} \le \frac{C}{|\tau|},$$

we obtain

$$\begin{aligned} \int_{-\infty}^{\infty}\psi^2(\cdot,\tau+1)z^{p-1}dx + \iint_{\Lambda_\tau}\eta(t)(\psi_x^2 + \psi^2)\,dx\,dt \\ \le C\Big(\|\psi\|^2_{L^2(\Lambda_\tau)} + \|g\|^2_{L^2(\Lambda_\tau)} \\ + \frac{1}{|\tau|}\sup_{t\in[\tau,\tau+1]}\Big(\int_{-\infty}^{\infty}\psi^2 z^{p-1}\,dx\Big)^{\frac12}\|\psi\|_{L^2(\Lambda_\tau)}\Big). \end{aligned} \tag{3.25}$$

If we now multiply (3.1) by $\psi_t(x,t)$, integrate by parts over $\mathbb{R}$, use the bound $\frac{|z_t|}{z} \le C|\dot\xi|$ and apply Cauchy–Schwarz to the last term, we obtain

$$\begin{aligned} \frac{p}{2}\int_{-\infty}^{\infty}\psi_t^2 z^{p-1}\,dx + \frac{1}{2}\frac{d}{dt}\Big(\int_{-\infty}^{\infty}(\psi_x^2 + \psi^2 - pz^{p-1}\psi^2)\,dx\Big) \\ \le C\Big(\int_{-\infty}^{\infty}(\psi^2 + g^2)z^{p-1}\,dx + |\dot\xi|\int_{-\infty}^{\infty}\psi^2 z^{p-1}\,dx\Big). \end{aligned} \tag{3.26}$$

Multiplying the last inequality by the cut off function $\eta(t)$ introduced above, integrating in time and using (3.25), we obtain (similarly as above) the bound

$$\begin{aligned}(3.27)\qquad \iint_{\Lambda_\tau} \eta(t)\psi_t^2 z^{p-1}\,dx\,dt + \int_{-\infty}^{\infty}(\psi_x^2+\psi^2-pz^{p-1}\psi^2)(\cdot,\tau+1)\,dx \\ \le C\Big(\|\psi\|^2_{L^2(\Lambda_\tau)}+\|g\|^2_{L^2(\Lambda_\tau)} \\ +\frac{1}{|\tau|}\sup_{t\in[\tau,\tau+1]}\Big(\int_{-\infty}^{\infty}\psi^2 z^{p-1}\,dx\Big)^{\frac12}\|\psi\|_{L^2(\Lambda_\tau)}\Big).\end{aligned}$$

Furthermore, (3.27), (3.25), the Sobolev embedding theorem in one dimension and the interpolation inequality yield the L^∞ estimate

$$\begin{aligned}\|\psi(\cdot,\tau+1)\|_{L^\infty(\mathbb{R})} &\le C\Big(\int_{-\infty}^{\infty}(\psi_x^2+\psi^2)(\cdot,\tau+1)\,dx\Big)^{\frac12}\\ &\le C\Big(\|\psi\|_{L^2(\Lambda_\tau)}+\|g\|_{L^2(\Lambda_\tau)}+\frac{1}{|\tau|}\|\psi\|_{L^\infty(\Lambda_\tau)}\Big).\end{aligned}$$

We next multiply the last inequality by $|\tau+1|^\nu$, for any $\nu\ge 0$. Since $s\le\tau\le t_0-1$, by choosing $|t_0|$ sufficiently large we obtain

$$|\tau+1|^\nu\|\psi\|_{L^\infty(\Lambda_{\tau+1})}\le C\big(\|\psi\|^\nu_{L^2_{s,t_0}}+\|g\|^\nu_{L^2_{s,t_0}}\big)+\frac12\sup_{\tau\in[s,t_0]}|\tau|^\nu\|\psi\|_{L^\infty(\Lambda_\tau)}.$$

Since $\tau+1\le t_0$ is arbitrary, we obtain

$$\sup_{\tau\in[s,t_0]}|\tau|^\nu\|\psi\|_{L^\infty(\Lambda_\tau)}\le C\big(\|\psi\|^\nu_{L^2_{s,t_0}}+\|g\|^\nu_{L^2_{s,t_0}}\big).$$

Since $\tau\in[s,t_0-1]$ is arbitrary, by choosing $|t_0|$ sufficiently large, we conclude

$$(3.28)\qquad \sup_{\tau\in[s,t_0]}|\tau|^\nu\|\psi\|_{L^\infty(\Lambda_\tau)}\le C\big(\|\psi\|^\nu_{L^2_{s,t_0}}+\|g\|^\nu_{L^2_{s,t_0}}\big).$$

In addition, integrating (3.26) on $[\tau,\tau+1]$ and using the previous estimates yields the bound

$$(3.29)\qquad \|\psi_t\|^\nu_{L^2_{s,t_0}}\le C\big(\|\psi\|^\nu_{L^2_{s,t_0}}+\|g\|^\nu_{L^2_{s,t_0}}\big).$$

Finally, from (3.29), (3.28) and equation (3.1) we obtain

$$\|z^{-(p-1)}(\psi_{xx}-\psi)\|^\nu_{L^2_{s,t_0}}\le C\big(\|\psi\|^\nu_{L^2_{s,t_0}}+\|g\|^\nu_{L^2_{s,t_0}}\big).$$

Combining the above estimates gives us the bound (3.23). □

We will proceed next to showing the same estimate as above for solutions of (3.11).

Lemma 3.3 (Energy H^2 and L^∞ estimate for equation (3.11)). *Let $\psi^s(x,t)$ be a solution to* (3.11). *Then there exists a $t_0<0$ so that for any $\nu\in[0,1)$ we have*

$$(3.30)\qquad \sup_{\tau\in[s,t_0]}|\tau|^\nu\|\psi\|_{L^\infty(\Lambda_\tau)}+\|\psi\|^\nu_{H^2_{s,t_0}}\le C\big(\|\psi\|^\nu_{L^2_{s,t_0}}+\|f\|^\nu_{L^2_{s,t_0}}\big).$$

Proof. If we apply the estimate from Lemma 3.2, with $g=f+C(\psi)$, we obtain

$$\sup_{\tau\in[s,t_0]}|\tau|^\nu\|\psi\|_{L^\infty(\Lambda_\tau)}+\|\psi\|^\nu_{H^2_{s,t_0}}\le C\big(\|\psi\|^\nu_{L^2_{s,t_0}}+\|C(\psi)\|^\nu_{L^2_{s,t_0}}+\|f\|^\nu_{L^2_{s,t_0}}\big).$$

On the other hand, it follows from (3.19) that

$$\|C(\psi)\|^{\nu}_{L^2_{s,t_0}} \le \frac{C}{\sqrt{|t_0|}} \|\psi\|^{\nu}_{H^2_{s,t_0}} \tag{3.31}$$

and the desired estimate follows by choosing t_0 so that $\frac{C}{\sqrt{|t_0|}} \le \frac{1}{2}$. □

Corollary 3.1 (Estimation of the error term). *Under the assumptions of Lemma* 3.1*, there exist uniform constants $t_0 < 0$ and $C > 0$ so that for any $\nu \in [0, 1)$, we have*

$$\|C(\psi)\|^{\nu}_{L^2_{s,t_0}} \le \frac{C}{\sqrt{|t_0|}} \big(\|\psi\|^{\nu}_{L^2_{s,t_0}} + \|f\|^{\nu}_{L^2_{s,t_0}}\big). \tag{3.32}$$

Proof. It readily follows from combining (3.31) and (3.30). □

We will next establish (3.21) as an a priori estimate.

Proposition 3.2. *There exist uniform constants $C < \infty$, $t_0 < 0$, such that if $\psi^s(x, t)$ is a solution of* (3.11) *with $s < \frac{3t_0}{2}$, which also satisfies the orthogonality conditions* (3.5)–(3.6)*, then*

$$\sup_{\tau \in [s,t_0]} |\tau|^{\nu} \|\psi^s(\cdot, \tau)\|_{L^2} \le C \|f\|_{L^2_{s,t_0}}. \tag{3.33}$$

It follows that (3.21) *holds.*

Proof. We will first establish estimate (3.33). We begin by observing that under the assumptions of the proposition, it will be sufficient to establish the bound

$$\sup_{s \le \tau \le t} \|\psi^s(\cdot, \tau)\|_{L^2} \le C \sup_{s \le \tau \le t} \|f\|_{L^2(\Lambda_\tau)} \tag{3.34}$$

for all t such that $s \le \frac{3t}{2} \le \frac{3t_0}{2}$, where we recall that $\Lambda_\tau = \mathbb{R} \times [\tau, \tau + 1]$. Indeed, if (3.34) holds, then for any t satisfying $s < \frac{3t}{2} \le \frac{3t_0}{2}$, we have

$$\begin{aligned}
|t|^{\nu} \|\psi^s(\cdot, t)\|_{L^2} &\le C |t|^{\nu} \sup_{s \le \tau \le t} \|f\|_{L^2(\Lambda_\tau)} \\
&\le \sup_{s \le \tau \le t} |\tau|^{\nu} \|f\|_{L^2(\Lambda_\tau)} \\
&\le \sup_{s \le \tau \le t_0} |\tau|^{\nu} \|f\|_{L^2(\Lambda_\tau)} \\
&= C \|f\|_{L^2_{s,t_0}}
\end{aligned}$$

which readily shows that (3.33) holds.

To establish the validity of (3.34) we argue by contradiction. If (3.34) does not hold, then there exist decreasing sequences $\bar{t}_k \to -\infty$ and $s_k < \frac{3\bar{t}_k}{2}$, $s_k \to -\infty$, and solutions ψ_k of the equation

$$p z^{p-1} \partial_t \psi_k = \partial_{xx} \psi_k - \psi_k + p z^{p-1} \psi_k + z^{p-1} [f_k - C(\psi_k, t)] \tag{3.35}$$

defined on $\mathbb{R} \times [s_k, \bar{t}_k]$, so that

$$\sup_{s_k \le \tau \le \bar{t}_k} \|\psi_k(\cdot, \tau)\|_{L^2} \ge k \sup_{s_k \le \tau \le \bar{t}_k} \|f_k\|_{L^2(\Lambda_\tau)}. \tag{3.36}$$

We may assume, without loss of generality, that

$$\sup_{s_k \le \tau \le \bar{t}_k} \int_{-\infty}^{\infty} \psi_k^2(x,\tau) z^{p-1}\, dx = 2 \tag{3.37}$$

otherwise we would perform the rescaling of the solutions for (3.37) to hold. Then, by (3.36), we have

$$\| f_k \|_{L^2_{s_k, \bar{t}_k}} \le \frac{C}{k}. \tag{3.38}$$

Because of (3.37), we can pick $t_k \in [s_k, \bar{t}_k]$ such that

$$\frac{3}{2} \le \int_{-\infty}^{\infty} \psi_k^2(x, t_k) z^{p-1}\, dx \le 2. \tag{3.39}$$

Also, passing to a subsequence if necessary, we may assume that t_k is decreasing.

Claim 3.2. *We have*

$$\liminf_{k \to \infty} (t_k - s_k) = +\infty.$$

Proof. We will apply (3.23) with $\psi = \psi_k$, $g = g_k := f_k - C(\psi_k, t)$ and for $\nu = 0$. To estimate the right-hand side of (3.23), we use (3.32), (3.38) and (3.37) to obtain for all $s_k \le t \le \bar{t}_k$ the bound

$$\|g_k(\cdot, t)\|_{L^2} \le \frac{C}{\sqrt{|\bar{t}_k|}} \left(\|\psi\|_{L^2_{s,\bar{t}_k}} + \frac{1}{k} \right) + \frac{C}{k} \le C \left(\frac{1}{\sqrt{|\bar{t}_k|}} + \frac{1}{k} \right).$$

Hence, for all $\tau \in [s_k, \bar{t}_k]$ we have

$$\int_{s_k}^{\tau} \int_{-\infty}^{\infty} g_k^2 z^{p-1}\, dx\, dt \le (\tau - s_k) \|g_k\|^2_{L^2_{\bar{t}_k}} \le C \left(\frac{1}{\sqrt{|\bar{t}_k|}} + \frac{1}{k} \right)^2 (\tau - s_k). \tag{3.40}$$

Set

$$\alpha(\tau) = \int_{s_k}^{\tau} \int_{-\infty}^{\infty} \psi^2 z^{p-1}\, dx\, dt.$$

It follows from (3.23) and the above discussion that $\alpha(\tau)$ satisfies the differential inequality

$$\alpha'(\tau) \le C\alpha(\tau) + \mu_k(\tau - s_k)$$

with

$$\mu_k = C \left(\frac{1}{\sqrt{|\bar{t}_k|}} + \frac{1}{k} \right)^2 \quad \text{and} \quad \alpha(s_k) = 0.$$

Solving this differential inequality gives

$$\alpha(\tau) \le \frac{\mu_k}{C^2} \left(e^{C(\tau - s_k)} - [1 + C(\tau - s_k)] \right) \le \frac{\mu_k}{C^2} e^{C(\tau - s_k)}$$

which combined with (3.23) and (3.40) gives the bound

$$\int_{-\infty}^{\infty} \psi^2(\cdot, \tau) z^{p-1}\, dx \le C_1 \mu_k e^{C(\tau - s_k)},$$

for all $\tau \in [s_k, \bar{t}_k]$, where C_1 is a different, but still uniform constant. Hence

$$\int_{-\infty}^{\infty} \psi^2(\cdot, \tau) z^{p-1}\, dx \le 1$$

as long as

$$e^{C(\tau - s_k)} < \frac{1}{C_1 \mu_k}.$$

Since $\int_{-\infty}^{\infty} \psi^2(\cdot, t_k) z^{p-1} dx \ge \frac{3}{2}$, we must have

$$e^{C(t_k - s_k)} \ge \frac{1}{C_1 \mu_k}.$$

Since $\lim_{k\to\infty} \mu_k = 0$, this readily implies the claim. □

Set

$$\bar{\psi}_k(x,t) = \psi_k(x, t+t_k) \quad \text{and} \quad \bar{f}_k(x,t) = f_k(x, t+t_k)$$

and observe that each $\bar{\psi}_k$ satisfies the equation

$$pz_k^{p-1} \partial_t \bar{\psi}_k = \partial_{xx} \bar{\psi}_k - \bar{\psi}_k + pz_k^{p-1} \bar{\psi}_k + z_k^{p-1} \big[\bar{f}_k - C_k(\bar{\psi}_k, t)) \big] \tag{3.41}$$

on $\mathbb{R} \times [\bar{s}_k, 0]$, with $\bar{s}_k := s_k - t_k$ and

$$z_k(x,t) := z(x, t+t_k), \quad \bar{z}_k(x,t) = \bar{z}(x, t+t_k)$$

and

$$C_k(\bar{\psi}_k, t) = d_1(\bar{\psi}_k, t+t_k) z_k(x,t) + d_2(\bar{\psi}_k, t+t_k) \bar{z}_k(x,t)$$

where $d_i(\bar{\psi}_k, t)$ are defined in terms of ψ_k as before. Notice that because of the previous claim, $\bar{s}_k \le -\sigma$, for a uniform constant σ. It follows from (3.30) (with $\nu = 0$) and (3.37) that $\bar{\psi}_k$ satisfy the bound

$$\|\bar{\psi}_k\|_{L^\infty(\mathbb{R}\times[\bar{s}_k,0])} + \|\bar{\psi}_k\|_{H^2_{\bar{s}_k,0}} \le C \tag{3.42}$$

for a uniform in k constant C.

Inequality (3.39) says that

$$\frac{3}{2} \le \int_{-\infty}^{\infty} \bar{\psi}_k^2(x,0) z_k^{p-1}\, dx \le 2. \tag{3.43}$$

If we integrate (3.24) in time on $[t_k - \delta, t_k]$ and use (3.43), we conclude, after a straightforward calculation, the bounds

$$1 \le \inf_{\tau\in[-\delta,0]} \int_{-\infty}^{\infty} \bar{\psi}_k^2(x,\tau) z_k^{p-1}\, dx \le 2 \tag{3.44}$$

for a uniform in k small constant $\delta > 0$.

Claim 3.3. *There exists a universal large constant $M > 0$ for which*

$$\sup_{\tau\in[-\delta,0]} \int_{-\xi(\tau+t_k)+M}^{\xi(\tau+t_k)-M} \bar{\psi}_k^2(x,\tau) z_k^{p-1}\, dx < \frac{1}{2}. \tag{3.45}$$

Proof. We recall that by (7.2),

$$\xi(\tau + t_k) = \frac{1}{2} \log(2b|\tau + t_k|) + O(1).$$

By symmetry ($\bar\psi_k$ is an even function), we only need to show that

$$\sup_{\tau\in[-\delta,0]}\int_0^{\xi(\tau+t_k)-M}\bar\psi_k^2(x,\tau)z_k^{p-1}\,dx<\frac{1}{4}.$$

Also, since for $x>0$ and $\tau\in[-\delta,0]$ we have

$$z_k(x,\tau)=w(x-\xi(\tau+t_k))+w(x+\xi(\tau+t_k))\le 2w(x-\xi(\tau+t_k)),$$

it will be enough to establish the inequality

$$\sup_{\tau\in[-\delta,0]}\int_0^{\xi(\tau+t_k)-M}\bar\psi_k^2(x,\tau)w^{p-1}(x-\xi(\tau+t_k))\,dx<\frac{1}{8}.$$

Using the L^∞ bound in (3.42), we conclude that for every $\tau\in[-\delta,0]$ we have

$$\int_0^{\xi(\tau+t_k)-M}\bar\psi_k^2(x,\tau)w^{p-1}(x-\xi(\tau+t_k))\,dx\le C\int_0^{\xi(\tau+t_k)-M}w^{p-1}(x-\xi(\tau+t_k))\,dx$$

for a uniform constant C. Finally, we have

$$\int_0^{\xi(\tau+t_k)-M}w^{p-1}(x-\xi(\tau+t_k))\,dx=\int_{-\xi(\tau+t_k)}^{-M}w^{p-1}(x)\,dx$$

where w is given by (1.9). It follows that there exists a uniform constant M such that

$$C\int_0^{\xi(\tau+t_k)-M}w^{p-1}(x-\xi(\tau+t_k))\,dx=C\int_{-\xi(\tau+t_k)}^{-M}w^{p-1}(x)\,dx<\frac{1}{8}$$

for all $\tau\in[-\delta,0]$ finishing the proof of the claim. □

We will now conclude the proof of the Proposition. By (3.44), (3.45) and the symmetry of $\bar\psi_k$, we have

$$\inf_{\tau\in[-\delta,0]}\int_{-\infty}^{\xi(\tau+t_k)+M}\bar\psi_k^2(x,\tau)z_k^{p-1}\,dx\ge\frac{1}{4}. \tag{3.46}$$

We wish to pass to the limit along a sub-sequence $k_l\to\infty$. However, in order that we see something nontrivial at the limit, we will need to perform a new change of variables defining

$$\phi_k(x,t):=\bar\psi_k(x-\xi(t+t_k),t),\quad t\le 0.$$

It follows that each ϕ_k satisfies the equation

$$pw_k^{p-1}\partial_t\phi_k=\partial_{xx}\phi_k-\phi_k+pw_k^{p-1}\phi_k-\dot\xi(t+t_k)\partial_x\phi_k+w_k^{p-1}g_k \tag{3.47}$$

on $-\bar s_k<t\le 0$ with $g_k(x,t):=\bar f_k(x-\xi(t+t_k),t)-C_k(\phi_k,t)$ and

$$w_k(x,t):=z_k(x-\xi(t+t_k),t)=w(x)+w(x-2\xi(t+t_k)).$$

Moreover, (3.38) and (3.32) imply the bounds

$$\|g_k\|_{L^2_{s,t_0}}\le C\left(\frac{1}{k}+\frac{1}{\sqrt{|t_k|}}\right) \tag{3.48}$$

and by (3.42) and the inequality $w_k \geq w$,

$$\|\phi_k\|_{L^\infty(\mathbb{R}\times[\bar{s}_k,0])} + \|\phi_k\|_{H^2_{\bar{s}_k,0}} \leq C. \tag{3.49}$$

In addition, (3.46) implies the following uniform bound:

$$\inf_{\tau\in[-\delta,0]} \int_{-\infty}^{M} \phi_k^2(x,\tau) w_k^{p-1}\, dx \geq \frac{1}{4}. \tag{3.50}$$

Set $Q = (-\infty, x_0] \times [\tau_0, 0]$, where $x_0 > 0$ is an arbitrary number and τ_0 is any number such that $\bar{s}_k < \tau_0 < 0$, for all k (recall that $\bar{s}_k \leq -\sigma$ for all k by Claim 3.2). It follows from the energy bound (3.49) that passing to a subsequence, still denoted by ϕ_k, we have $\phi_k \to \phi$ in $L^2_w(Q)$ and $\phi_k \to \phi$ weakly in $H^1_w(Q)$. Passing to the limit in (3.47) while using (3.48) and the bound

$$\int_{\tau_0}^{0} \dot{\xi}^2(t+t_k)\, dt = O\Big(\frac{1}{|t_k|^2}\Big),$$

we conclude that ϕ is a weak solution of

$$p w^{p-1} \partial_t \phi = \partial_{xx}\phi - \phi + p w^{p-1}\phi \tag{3.51}$$

on $\mathbb{R} \times (-\infty, 0)$. Standard regularity theory shows that ϕ is actually a smooth solution. In addition, ϕ satisfies the orthogonality conditions

$$\int_{-\infty}^{\infty} \phi(x,t) w'(x) w^{p-1}(x)\, dx = 0 \quad \text{for a.e. } t < t_0 \tag{3.52}$$

and

$$\int_{-\infty}^{\infty} \phi(x,t) w(x) w^{p-1}(x)\, dx = 0 \quad \text{for a.e. } t < t_0. \tag{3.53}$$

Moreover, from (3.42) we have the following uniform estimate:

$$\sup_{\tau\leq 0} \int_{\tau}^{\tau+1} \int_{-\infty}^{\infty} \phi^2(x,t) w^{p-1}\, dx\, dt \leq 2. \tag{3.54}$$

Also, passing to the limit in (3.50) we conclude that

$$\int_{-\delta}^{0} \int_{-\infty}^{M} \phi^2(x,t) w^{p-1}\, dx\, dt \geq \frac{\delta}{4} > 0$$

which shows that our limit ϕ is nontrivial. From Claim 3.2 we have $\liminf_{k\to\infty} \bar{s}_k = -\infty$. Hence, we may assume, passing to a subsequence, that $\bar{s}_k \to -\infty$. It follows, that the limit ϕ is an ancient solution of equation (3.51), i.e. defined on $\mathbb{R} \times (-\infty, 0]$ which satisfies the orthogonality conditions (3.52) and (3.53).

Set $\alpha(t) = \frac{1}{2}\|\phi(\cdot,t)\|_{L^2(w^{p-1}\,dx)}$ and observe that since ϕ is orthogonal to the two eigenfunctions of the operator L_0 (defined in (3.3)) corresponding to its only two nonnegative eigenvalues λ_{-1} and λ_0, we have

$$\alpha'(t) \leq -\lambda\alpha(t), \quad t \leq 0,$$

for some $\lambda > 0$, implying that $\alpha(t) \geq \alpha(0)e^{\lambda|t|}$ which contradicts (3.54). This finishes the proof of the proposition. □

3.2. The proofs of Lemma 3.1 and Proposition 3.1. Based on the a priori estimates of the previous subsection, we will give now the proofs of Lemma 3.1 and Proposition 3.1.

Proof of Lemma 3.1. It will be sufficient to establish the existence of a solution ψ^s to (3.11). Indeed, given the existence of solution ψ^s, by the fact that the forcing term f satisfies orthogonality conditions (3.5) and (3.6) we already know that $\psi^s(\cdot,t)$ will continue to satisfy conditions (3.5) and (3.6) for $t \geq s$. Then estimate (3.21) follows by Proposition 3.2.

The strategy for establishing the existence of ψ^s is as follows. Fix an $s < t_0 - 1$. We first establish the existence of a solution ψ^s to the initial value problem (3.22) on $\mathbb{R} \times [s, s+\tau_0]$, for a given function g with

$$(*) \qquad \|g\|^{\nu}_{L^2_{t_0}} < \infty,$$

where τ_0 is a uniform constant, independent of s, to be chosen below. Then we solve the nonlocal problem (3.11) on $\mathbb{R} \times [s, s+\tau_0]$. At the end we show how to extend such a solution in time up to t_0, to obtain a solution of (3.11).

We first claim that given an $s < t_0 - 1$ and a function g with $(*)$, there exists a solution ψ^s of (3.22), on $\mathbb{R} \times [s, s+\tau_0]$, for some τ_0 to be chosen below. The solution ψ^s will be constructed as the limit, as $R \to \infty$, of solutions ψ^s_R to the Dirichlet problems

$$(3.55) \qquad \begin{cases} pz^{p-1}\partial_t \psi^s_R = (\psi^s_R)_{xx} - \psi^s_R + pz^{p-1}\psi^s_R + z^{p-1}g & \text{on } Q_{R,s}, \\ \psi^s_R(\cdot, s) = 0 & \text{on } \partial_p Q_{R,s}, \end{cases}$$

on $Q_{R,s} := [-R, R] \times [s, s+\tau_0]$. Since our weight z^{p-1} is bounded from above and below away from zero on $Q_{R,s}$, by standard parabolic theory there exists a solution ψ^s_R to the same Dirichlet problem on the set $\hat{Q}_{R,s} := [-R, R] \times [s, s+\tau_R]$, for some $\tau_R > 0$ which will be taken to satisfy $\tau_R \leq 1$. Similarly as in the proof of Lemma 3.2, $\psi := \psi^s_R$ satisfies the estimate

$$(3.56) \qquad \begin{aligned} &\frac{d}{dt}\int_{-R}^{R} \psi^2 z^{p-1}\,dx + \int_{-R}^{R} (\psi^2 + \psi_x^2)\,dx \\ &\quad \leq C_1\Big(\int_{-R}^{R} g^2 z^{p-1}\,dx + (\dot\xi + 1)\int_{-R}^{R} \psi^2 z^{p-1}\,dx\Big) \end{aligned}$$

for a universal constant C_1. Before we integrate (3.56) in time, we observe that

$$(3.57) \qquad \begin{aligned} \int_s^{s+\tau_R} \dot\xi \int_{-R}^{R} \psi^2 z^{p-1}\,dx\,dt &\leq \Big(\int_s^{s+\tau_R} \dot\xi^2\,dt\Big)^{\frac12} \sup_{[s,s+\tau_R]} \int_{-R}^{R} \psi^2 z^{p-1}\,dx \\ &\leq \epsilon \sup_{[s,s+\tau_R]} \int_{-R}^{R} \psi^2 z^{p-1}\,dx \end{aligned}$$

if s is chosen sufficiently close to $-\infty$. The last inequality follows from the fact that

$$\dot\xi(t) = \frac{1}{2|t|} + \dot h(t) \quad \text{and} \quad \|h\|^{\mu,1+\mu}_{1,\sigma,t_0} \leq 1$$

by assumption.

Set

$$\tau_0 := \min\Big\{\frac{1}{3C_1}, 1\Big\},$$

with C_1 being the constant from (3.56) and take $\tau_R \le \tau_0$. Integrating in time (3.56), while using the Dirichlet boundary condition in (3.55), the Cauchy–Schwarz inequality and (3.57) with ϵ chosen sufficiently small, we obtain

$$\begin{aligned}\frac{2}{3}\sup_{[s,s+\tau_R]}\int_{-R}^{R}\psi^2 z^{p-1}\,dx &+ \iint_{\hat{Q}_{R,s}}\Big(\psi_x^2+\frac{1}{2}\psi^2\Big)\,dx\,dt\\ &\le C\iint_{\hat{Q}_{R,s}} g^2 z^{p-1}dxdt + C_1\iint_{\hat{Q}_{R,s}}\psi^2 z^{p-1}dx\,dt\\ &\le C\iint_{\hat{Q}_{R,s}} g^2 z^{p-1}\,dx\,dt + \frac{1}{3}\sup_{[s,s+\tau_R]}\int_{-R}^{R}\psi^2\,dx\,dt\end{aligned}$$

since $C_1|\tau_R| \le C_1\tau_0 \le \frac{1}{3}$ by our choice of C_1. It follows that $\psi := \psi_R^s$ satisfies

$$\begin{aligned}\sup_{[s,s+\tau)}\int_{-R}^{R}\psi^2 z^{p-1}\,dx &+ \iint_{Q_{R,s}}\Big(\psi_x^2+\frac{1}{2}\psi^2\Big)\,dx\,dt\\ &\le C_0\iint_{Q_{R,s}} g^2 z^{p-1}\,dx\,dt\end{aligned}\tag{3.58}$$

for a uniform constant C_0. Similarly to deriving the energy estimate in Lemma 3.2, using (3.58) and the fact that $(\psi_R^s)_x(\cdot,s)=0$, we find that $\psi := \psi_R^s$ also satisfies

$$\begin{aligned}\iint_{\hat{Q}_{R,s}}\psi_t^2 z^{p-1}\,dx\,dt &+ \frac{1}{2}\sup_{[s,s+\tau_R]}\int_{-R}^{R}(\psi^2+\psi_x^2)\,dx\\ &\le C_0\iint_{\hat{Q}_{R,s}} g^2 z^{p-1}\,dx\,dt\end{aligned}\tag{3.59}$$

where C_0 is a constant, possibly larger than the constant in (3.58), but still independent of R, s. The right-hand sides in both inequalities (3.58) and (3.59) are bounded by a constant that is independent of R, namely

$$C_0\int_s^{s+1}\int_{-\infty}^{\infty} g^2 z^{p-1}\,dx\,dt.$$

Hence, by standard linear parabolic theory the solution ψ_R^s will exist at least for $s \le t \le \tau_0$, namely on $Q_{R,s}$. Take a sequence $R_j \to +\infty$ and set $\Lambda_{s,\tau_0} := \mathbb{R}\times[s,s+\tau_0]$. Since the equation in (3.22) is nondegenerate on any compact subset K of Λ_{s,τ_0}, the uniform estimates (3.58)–(3.59) and standard arguments imply that a subsequence of solutions $\psi_{R_j}^s$ converges in $C^\infty(K)$ to a smooth solution ψ^s of problem (3.22). The limiting smooth solution ψ^s is defined on Λ_{s,τ_0}.

The next step is to show that we can solve a nonlocal problem (3.11) on Λ_{s,τ_0}. We will do that via contraction mapping arguments. Define a set

$$X^s := \{\psi : \|\psi\|_{H^2(\Lambda_s)} < \infty\}.$$

We consider the operator $A^s : X^s \to X^s$ given by

$$A^s(\psi) := T^s(f - C(\psi))$$

where $T^s(g)$ denotes the solution to (3.22) constructed above and

$$C(\psi) = d_1 z + d_2\bar{z}$$

where (d_1, d_2) is the unique solution of the linear system (3.13).

We will show that the map A^s defines a contraction mapping and we will apply the fixed point theorem to it. To this end, set $c := C_0\|f\|_{L^2(\Lambda_s)}$ and $X^s_c := \{\psi \in X^s : \|\psi\|_{H^2(\Lambda_s)} < 2c\}$, where the constant C_0 is taken from (3.58)–(3.59). We claim that $A^s(X^s_c) \subset X^s_c$. To show this claim, let $\psi \in X^s_c$. Estimates (3.58), (3.59), estimate (3.19) for $C(\psi)$ and the Sobolev embedding yield

$$\begin{aligned}\|A^s(\psi)\|_{H^2(\Lambda_s)} &= \|T^s(f - C(\psi))\|_{H^2(\Lambda_s)} \\ &\le C_0\|f - C(\psi)\|_{L^2(\Lambda_s)} \\ &\le C_0(\|f\|_{L^2(\Lambda_s)} + \|C(\psi)\|_{L^2(\Lambda_s)}) \\ &= c + C_0\|C(\psi)\|_{L^2(\Lambda_s)} \\ &\le c + \frac{C}{\sqrt{|s|}}\|\psi\|_{L^2(\Lambda_s)} \\ &< 2c\end{aligned} \tag{3.60}$$

if $|s|$ sufficiently large (which holds if t_0 is chosen sufficiently close to $-\infty$). Next we show that A^s defines a contraction map. Indeed, since $C(\psi)$ is linear in ψ, we have

$$\begin{aligned}\|A^s(\psi_1) - A^s(\psi_2)\|_{H^2(\Lambda_s)} &= \|T^s(C(\psi_1) - C(\psi_2))\|_{H^2(\Lambda_s)} \\ &\le C_0\|C(\psi_1) - C(\psi_2)\|_{L^2(\Lambda_s)} \\ &= C_0\|C(\psi_1 - \psi_2)\|_{L^2(\Lambda_s)} \\ &\le \frac{C}{\sqrt{|s|}}\|\psi_1 - \psi_2\|_{H^2(\Lambda_s)} \\ &\le \frac{1}{2}\|\psi_1 - \psi_2\|_{H^2(\Lambda_s)}.\end{aligned} \tag{3.61}$$

By estimates (3.60)–(3.61), the fixed point theorem implies that there exists a $\psi^s \in X^s$ so that $A^s(\psi^s) = \psi^s$, meaning that equation (3.11) has a solution ψ^s, defined on Λ_{s,τ_0}.

We claim that $\psi^s(\cdot, t)$ can be extended to a solution on $\mathbb{R} \times [s, t_0]$, still satisfying our orthogonality conditions and a priori estimates. To this end, assume that our solution $\psi^s(\cdot, t)$ exists for $s \le t < T$, where $T < t_0$ is the maximal time of existence. Since $\psi^s(\cdot, t)$ satisfies the orthogonality conditions (3.5) and (3.6) for $t \in [s, T]$, by Proposition 3.2,

$$\sup_{t\in[s,T)} |\tau|^\nu \|\psi^s(\cdot, t)\|_{L^2(\mathbb{R})} \le C\|f\|_{L^2_{s,T}} \tag{3.62}$$

and

$$\sup_{t\in[s,T)} |\tau|^\nu \|\psi^s(\cdot, t)\|_{L^\infty(\mathbb{R})} + \|\psi\|_{H^2_{s,T}} \le C\|f\|_{L^2_{s,T}} \tag{3.63}$$

where C is a uniform constant. Since

$$\|f\|_{L^2_{s,T}} \le \|f\|_{L^2_{s,t_0}} \le C,$$

it follows that ψ^s can be extended past time T, unless $T = t_0$. Moreover, (3.21) is satisfied as well and ψ^s also satisfies the orthogonality conditions. □

Having Lemma 3.1 we are able to conclude the proof of Proposition 3.1.

Proof of Proposition 3.1. Having Lemma 3.1 we are able to conclude the proof of the proposition. Take a sequence of $s_j \to -\infty$. By Lemma 3.1, for every s_j there is a solution ψ^{s_j} to equation (3.11) such that $\psi^{s_j}(\cdot, s_j) = 0$ and it satisfies the uniform estimate (3.21), independent of s_j. Moreover, our equation (3.11) is nondegenerate on every compact subset $K \subset \mathbb{R} \times (-\infty, t_0)$. Therefore on K we can apply standard parabolic theory to get higher order derivative estimates for our sequence of solutions ψ^{s_j}, which are independent of s_j but may depend on K. Let $j \to \infty$. By the Arzela–Ascoli theorem and a standard diagonalization argument we conclude that a subsequence $\{\psi^{s_j}\}$ converges, as $j \to \infty$, to a smooth function ψ defined on $\mathbb{R} \times (-\infty, t_0)$. Moreover, ψ satisfies the orthogonality conditions (3.5) and (3.6) and(!) estimate (3.20). The latter follows from taking the limit as $j \to \infty$ in (3.62) and (3.63), both satisfied by ψ^{s_j}, and the fact that the constants on the right-hand side are independent of j. □

3.3. $W^{2,\sigma}$ estimates. We will next derive weighted $W^{2,\sigma}$ estimates for the linear equation (3.9). We recall that the $W^{2,\sigma}$ norm is given by Definitions 2.3 and 2.4. We have the following global estimate.

Proposition 3.3. *Let ψ be a solution of* (3.9) *as in Proposition* 3.1. *If $\|f\|^{\nu}_{\sigma,t_0} < \infty$ for some $\sigma > 2$ and $\nu \in [0,1]$, then*

$$\|\psi\|^{\nu}_{2,\sigma,t_0} \le C\big(\|f\|^{\nu}_{L^2_{t_0}} + \|f\|^{\nu}_{\sigma,t_0}\big). \tag{3.64}$$

The proof of Proposition 3.3 will follow from a similar a priori estimate for solutions of (3.1).

Lemma 3.4. *Let ψ be an even solution of equation* (3.1) *with g a given even function that satisfies $\|g\|^{\nu}_{\sigma,t_0} + \|g\|^{\nu}_{L^2_{t_0}} < \infty$ for some $\sigma > 2$ and $\nu \in [0,1)$. Then, we have*

$$\|\psi\|^{\nu}_{2,\sigma,t_0} \le C\big(\|\psi\|^{\nu}_{L^2_{t_0}} + \|g\|^{\nu}_{L^2_{t_0}} + \|g\|^{\nu}_{\sigma,t_0}\big). \tag{3.65}$$

Before we give the proof of Lemma 3.4, we will prove Proposition 3.3 using Lemma 3.4.

Proof of Proposition 3.3. Assume that ψ is a solution of equation (3.9), as in the statement of the proposition. It follows from Lemma 3.4 and the L^2 estimate in Proposition 3.1 that

$$\|\psi\|^{\nu}_{2,\sigma,t_0} \le C\big(\|f\|^{\nu}_{L^2_{t_0}} + \|f\|^{\nu}_{\sigma,t_0} + \|C(\psi)\|^{\nu}_{L^2_{t_0}} + \|C(\psi)\|^{\nu}_{\sigma,t_0}\big).$$

It follows from (3.31) and the estimate in Proposition 3.1 that

$$\|C(\psi)\|^{\nu}_{L^2_{t_0}} \le \frac{C}{\sqrt{|t_0|}}\|f\|^{\nu}_{L^2_{t_0}}. \tag{3.66}$$

In addition, it can be shown, similarly as in the proof of (3.31), that

$$\|C(\psi)\|^{\nu}_{\sigma,t_0} \le \frac{C}{\sqrt{|t_0|}}\|\psi\|^{\nu}_{2,\sigma,t_0}. \tag{3.67}$$

Combining the last three estimates, readily yields the estimate of the proposition, provided that $|t_0|$ is chosen sufficiently large. □

Before we proceed with the proof of Lemma 3.4, let us summarize the estimate we have for $C(\psi)$, using Propositions 3.1 and 3.3.

Corollary 3.2. *Under the assumptions of Proposition* 3.3 *we have*

$$\|C(\psi)\|^{\nu}_{*,\sigma,t_0} \le \frac{C}{\sqrt{|\tau_0|}}\big(\|\psi\|^{\nu}_{H^2_{t_0}} + \|\psi\|^{\nu}_{2,\sigma,t_0}\big). \tag{3.68}$$

It follows that

$$\|C(\psi)\|^{\nu}_{*,\sigma,t_0} \le \frac{C}{\sqrt{|\tau_0|}}\|f\|^{\nu}_{*,\sigma,t_0} \tag{3.69}$$

for a universal constant C.

Proof. Estimate (3.68) readily follows by combining (3.31) and (3.67). Estimate (3.69) follows from (3.68) and the bounds in Propositions 3.1 and 3.3. □

We will now proceed to the proof of Lemma 3.4.

Proof of Lemma 3.4. We first observe that since both z and ψ are even functions in x, we will only need to establish the lemma on $-\infty < x \le 0$. We first perform a translation in space, setting

$$\phi(x,t) = \psi(x - \xi_0(t), t), \quad -\infty < x < \xi_0(t),$$

where $\xi_0(t) = \frac{1}{2}\log(2b|t|)$. It follows that ϕ satisfies the equation

$$p\bar{z}^{p-1}\partial_t\phi = \partial_{xx}\phi - p\dot{\xi}_0\bar{z}^{p-1}\partial_x\phi - \phi + p\bar{z}^{p-1}\phi + \bar{z}^{p-1}\bar{g} \tag{3.70}$$

with

$$\bar{z}(x,t) := w(x+\xi(t)-\xi_0(t)) + w(x-\xi(t)-\xi_0(t)), \quad \bar{g}(x,t) := g(x-\xi_0(t),t).$$

We observe that on the interval of consideration $-\infty < x \le \xi_0(t)$, we have

$$w(x-\xi(t)-\xi_0(t)) \le w(x+\xi(t)-\xi_0(t)),$$

hence

$$w(x+\xi(t)-\xi_0(t)) \le z(x,t) \le 2w(x+\xi(t)-\xi_0(t)) \quad \text{on } -\infty < x \le \xi_0(t). \tag{3.71}$$

If we divide equation (3.70) by $\bar{z}^{p-1}$ and perform the change of variables

$$\phi(x,t) = e^x\tilde{\phi}(r,t), \quad r = e^{\beta x}, \tag{3.72}$$

we conclude, after a simple calculation, that the new function $\tilde{\phi}(r,t)$ satisfies the equation

$$\tilde{\phi}_t = \alpha(r,t)\Delta\tilde{\phi} - \beta\dot{\xi}_0 r\tilde{\phi}_r + (1-\dot{\xi}_0)\tilde{\phi} + \tilde{g}(r,t) \tag{3.73}$$

with

$$\alpha(r,t) := p^{-1}\beta^2 U^{1-p}(r,t)$$

and

$$U(r,t) := e^{-x}(w(x+\xi(t)-\xi_0(t)) + w(x-\xi(t)-\xi_0(t))), \quad r = e^{\beta x}.$$

To obtain (3.73), we compute directly that

$$\phi_{xx} - \phi = \beta^2 e^{(1+2\beta)x}\left(\tilde{\phi}_{rr} + \frac{n-1}{r}\tilde{\phi}_r\right)$$

and

$$p\bar{z}^{p-1}\phi_t = pe^x\tilde{\phi}_t(e^x U)^{p-1} = pe^{(1+2\beta)x}U^{p-1}\tilde{\phi}_t$$

and similarly

$$p\bar{z}^{p-1}\bar{g} = pe^{(1+2\beta)x}U^{p-1}\tilde{g}.$$

Combining the last three equalities, we readily conclude (3.73). In addition, it follows from (1.9) and (3.71) that

$$\left(\frac{2k_n e^{\beta(\xi_0(t)-\xi(t))}}{e^{2\beta(\xi_0(t)-\xi(t))}+r^2}\right)^{\frac{1}{\beta}} \le U(r,t) \le 2\left(\frac{2k_n e^{\beta(\xi_0(t)-\xi(t))}}{e^{2\beta(\xi_0(t)-\xi(t))}+r^2}\right)^{\frac{1}{\beta}}. \tag{3.74}$$

Observe that

$$|\xi_0(t) - \xi(t)| = |h(t)| \le C|t|^{-\mu},$$

since $\|h\|^{\mu,1+\mu}_{1,\sigma,t_0} < \infty$ by assumption. This together with (3.74), the fact that $p-1 = \frac{4}{n-2} = 2\beta$, imply the estimate for the ellipticity coefficient $\alpha(r,t)$,

$$d_1\left(\frac{1}{2}+r^2\right)^2 \le \alpha(r,t) \le d_2(1+r^2)^2$$

for d_1 and d_2 universal positive constants.

We fix $\tau \le t_0$. We will next establish sharp $W^{2,\sigma}$ estimates for (3.73) on $B_{R(\tau)}\times[\tau-2,\tau]$, where $R(\tau) := e^{\beta\xi_0(\tau)}$ is a large number. Let $\bar{Q} := B_2 \times [\tau-2,\tau]$ and $Q := B_1 \times [\tau-1,\tau]$. By the standard parabolic $W^{2,\sigma}$, we have

$$\|\tilde{\phi}\|_{W^{2,\sigma}(Q)} \le C\left(\|\tilde{\phi}\|_{L^\sigma(\bar{Q})} + \|\tilde{g}\|_{L^\sigma(\bar{Q})}\right). \tag{3.75}$$

Translating this estimate back to the original coordinates and in terms of ψ gives us the desired weighted $W^{2,\sigma}$ bound on the exterior region, namely

$$\|\psi\|_{2,\sigma,E_\tau} \le C\left(\|\psi\|_{\sigma,\bar{E}_\tau} + \|g\|_{\sigma,\bar{E}_\tau}\right) \tag{3.76}$$

where $E_\tau = (-\infty, -\xi_0(\tau)) \times [\tau-1,\tau]$ and $\bar{E}_\tau = (-\infty, -\xi_0(\tau) + \frac{\ln 2}{\beta}) \times [\tau-2,\tau]$.

We will next obtain a weighted $W^{2,\sigma}$ estimate on $B_{R(\tau)} \setminus B_1$. To this end, we will assume that $R(\tau) = 2^{k_0}$ for a large constant $k_0 = k_0(\tau)$ and we will derive the estimate on the annuli

$$\{2^k < |x| < 2^{k+1}\} \times [\tau-1,\tau] \quad \text{for any } k = 0,\dots,k_0-1.$$

Set $\rho = 2^k$, $D_\rho = \{\rho < r < 2\rho\}\times[\tau-1,\tau]$ and $\bar{D}_\rho = \{\frac{\rho}{2} < r < 4\rho\}\times[\tau-2,\tau]$. Then, on $\bar{D}_\rho$ we have

$$\lambda\rho^4 \le \alpha(r,t) \le \Lambda\rho^4$$

for $\lambda > 0$ and $\Lambda < \infty$ universal constants. We will then divide the time interval $[\tau-1,\tau]$ into subintervals of length $\frac{1}{\rho^2}$ and in each of them we scale our solution $\tilde{\phi}$ to make equation (3.73) strictly parabolic. Let us denote by $[s-\frac{1}{\rho^2}, s]$ one such sub-interval and consider the cylindrical regions $D_\rho^s = \{\rho < r < 2\rho\} \times [s - \frac{1}{\rho^2}, s]$ and $\bar{D}_\rho^s = \{\frac{\rho}{2} < r < 4\rho\} \times [s - \frac{2}{\rho^2}, s]$. It follows that the rescaled solution

$$\phi_\rho(r,t) := \tilde{\phi}(\rho r, s + \rho^{-2} t), \quad (r,t) \in \bar{D} := \left\{\frac{1}{2} < r < 4\right\} \times [\tau - 2, \tau],$$

satisfies the equation

$$\partial_t \phi_\rho = \frac{\alpha(r,t)}{\rho^4} \Delta\phi_\rho - \beta\dot{\xi}_0(t)\frac{r}{\rho^3}\partial_r\phi_\rho + \frac{1}{\rho^2}(1 - \dot{\xi}_0(t))\phi_\rho + \frac{1}{\rho^2} f_\rho(r,t) \tag{3.77}$$

on $\bar{D}$ with $g_\rho(r,t) := \tilde{g}(\rho r, s + \rho^{-2} t)$. Moreover,

$$\lambda \le \frac{\alpha(r,t)}{\rho^4} \le \Lambda.$$

Recall also that $\dot{\xi}_0(t) = \frac{1}{t}$, and in particular, $|\dot{\xi}_0|$ is bounded. Hence, by the standard $W^{2,\sigma}$ estimates on equation (3.77), we have

$$\|\phi_\rho\|_{L^{2,\sigma}(D)} \le C\big(\|\phi_\rho\|_{L^\sigma(\bar{D})} + \rho^{-2}\|g_\rho\|_{L^\sigma(\bar{D})}\big)$$

with $D := \{1 < r < 2\} \times [\tau - 1, \tau]$ and $\bar{D} := \{\frac{1}{2} < r < 4\} \times [\tau - 2, \tau]$. This readily yields the bound

$$\begin{aligned}\|\tilde{\phi}_t\|_{L^\sigma(D_\rho^s)} + \rho^4\|D^2\tilde{\phi}\|_{L^\sigma(D_\rho^s)} + \rho^3\|D\tilde{\phi}\|_{L^\sigma(D_\rho^s)} + \rho^2\|\tilde{\phi}\|_{L^\sigma(D_\rho^s)}\\ \le C\big(\rho^2\|\tilde{\phi}\|_{L^\sigma(\bar{D}_\rho^s)} + \|\tilde{g}\|_{L^\sigma(\bar{D}_\rho^s)}\big).\end{aligned}$$

By repeating the above estimate on all time sub-intervals, we finally conclude

$$\begin{aligned}\|\tilde{\phi}_t\|_{L^\sigma(D_\rho)} + \rho^4\|D^2\tilde{\phi}\|_{L^\sigma(D_\rho)} + \rho^3\|D\tilde{\phi}\|_{L^\sigma(D_\rho)} + \rho^2\|\tilde{\phi}\|_{L^\sigma(D_\rho)}\\ \le C\big(\rho^2\|\tilde{\phi}\|_{L^\sigma(\bar{D}_\rho)} + \|\tilde{g}\|_{L^\sigma(\bar{D}_\rho)}\big).\end{aligned} \tag{3.78}$$

Because the first terms on the left-hand side of (3.78) have a growth in ρ, we will need to weight the L^σ norms by a power r^λ, for some appropriate $\lambda < 0$ to be chosen in the sequel. To this end, we define for any function $\tilde{h}$ the norm

$$\|\tilde{h}\|_{L^\sigma_\lambda(A)} = \left(\iint_A |\tilde{h}|^\sigma r^{\lambda+n-1}\, dr\, dt\right)^{\frac{1}{\sigma}}$$

and observe that (3.78) readily implies the following estimate in the new norms:

$$\begin{aligned}\|\tilde{\phi}_t\|_{L^\sigma_\lambda(D_\rho)} + \|r^4 D^2\tilde{\phi}\|_{L^\sigma_\lambda(D_\rho)} + \|r^3 D\tilde{\phi}\|_{L^\sigma_\lambda(D_\rho)} + \|r^2\tilde{\phi}\|_{L^\sigma_\lambda(D_\rho)}\\ \le C\big(\|r^2\tilde{\phi}\|_{L^\sigma_\lambda(\bar{D}_\rho)} + \|\tilde{g}\|_{L^\sigma_\lambda(\bar{D}_\tau)}\big).\end{aligned} \tag{3.79}$$

We will use the above local estimate to establish an estimate on the entire inner region. To this end, set $D_\tau = \{1 < r < R(\tau)\} \times [\tau - 1, \tau]$ and $\bar{D}_\tau = \{\frac{1}{2} < r < 2R(\tau)\} \times [\tau - 2, \tau]$, where $R(\tau) := e^{2\beta\xi_0(\tau)}$, as before. Applying (3.79) for all $\rho = 2^k$, $k = 0, 1, \dots, k_0$, where $R(\tau) = 2^{k_0}$, we obtain the bound

$$\begin{aligned}\|\tilde{\phi}_t\|_{L^\sigma_\lambda(D_\tau)} + \|r^4 D^2\tilde{\phi}\|_{L^\sigma_\lambda(D_\tau)} + \|r^3 D\tilde{\phi}\|_{L^\sigma_\lambda(D_\tau)} + \|r^2\tilde{\phi}\|_{L^\sigma_\lambda(D_\tau)}\\ \le C\big(\|r^2\tilde{\phi}\|_{L^\sigma_\lambda(\bar{D}_\tau)} + \|\tilde{g}\|_{L^\sigma_\lambda(\bar{D}_\tau)}\big).\end{aligned} \tag{3.80}$$

Before we find the appropriate λ, we will express the bound (3.80) back in terms of the functions $\phi(x,t) = e^x\tilde{\phi}(r,t)$ and $f(x,t) = e^x\tilde{f}(r,t)$ through the change of variables $r = e^{\beta x}$. Let $I_\tau := [0, \xi_0(\tau)] \times [\tau - 1, \tau]$ and $\bar{I}(\tau) := [-\frac{\ln 2}{\beta}, \xi_0(\tau) + \frac{\ln 4}{\beta}] \times [\tau - 2, \tau]$ denote the images of the sets D_τ and $\bar{D}_\tau$ under this change of variables. We will use the formula

$$\frac{\partial}{\partial r} = (\beta r)^{-1} \frac{\partial}{\partial x}$$

and the bounds

$$ce^{-\beta x} \le w^\beta(x) \le Ce^{-\beta x} \tag{3.81}$$

which hold in the region of consideration. A direct calculation shows that (3.80) implies the bound

$$\|\partial_t \phi\|_{L^\sigma_\lambda(I_\tau)} + \|\partial_x^2 \phi\|_{L^\sigma_\lambda(I_\tau)} + \|\partial_x \phi\|_{L^\sigma_\lambda(I_\tau)} \le C\big(\|e^{2\beta x}\phi\|_{L^\sigma_\lambda(\bar{I}_\tau)} + \|\bar{g}\|_{L^\sigma_\lambda(\bar{I}_\tau)}\big) \tag{3.82}$$

where, for any function h and any $I \subset [0,\infty) \times (-\infty, 0]$, we denote

$$\|h\|_{L^\sigma_\lambda(I)} := \left(\iint_I |h|^\sigma e^{(\lambda+n)\beta x}\, dx\, dt\right)^{\frac{1}{\sigma}}.$$

We next observe that the same arguments as in the proof of Lemma 3.2 give us a global L^∞ bound on the solution ψ of (3.9), namely

$$\|\psi\|_{L^\infty(\mathbb{R}\times(-\infty,t_0])} \le C\big(\|\psi\|_{L^2_{t_0}} + \|g\|_{L^2_{t_0}}\big)$$

which gives a similar bound for ϕ, namely

$$\|\phi\|_{L^\infty(\mathbb{R}\times(-\infty,t_0])} \le C\big(\|\phi\|_{L^2_{t_0}} + \|\bar{g}\|_{L^2_{t_0}}\big)$$

where

$$\|\bar{g}\|_{L^2_{t_0}} := \sup_{\tau \le t_0} \left(\iint_{\Lambda_\tau} \bar{g}^2 \bar{z}^{p-1}\, dx\, dt\right)^{\frac{1}{2}}.$$

Using this bound we obtain

$$\|e^{2\beta x}\phi\|_{L^\sigma_\lambda(\bar{I}_\tau)} \le C\big(\|\bar{g}\|_{L^2_{t_0}} + \|\phi\|_{L^2_{t_0}}\big)\left(\iint_{\bar{I}_\tau} e^{(2\sigma+\lambda+n)\beta x}\, dx\, dt\right)^{\frac{1}{\sigma}}.$$

The last integral is bounded uniformly in τ if λ is chosen so that

$$2\sigma + \lambda + n < 0.$$

Choose $\lambda = -(2\sigma + n + \theta)$, with $\theta > 0$ any small universal constant. With this choice of λ and for any function h we have

$$\|h\|_{L^\sigma_\lambda(I)} = \left(\iint_I |h|^\sigma e^{(\lambda+n)\beta x}\, dx\, dt\right)^{\frac{1}{\sigma}} = \left(\iint_I |h|^\sigma e^{-(2\sigma+\theta)\beta x}\, dx\, dt\right)^{\frac{1}{\sigma}}.$$

With such a choice of λ, combining this last estimate with (3.82), yields the bound

$$\|\phi\|_{2,\sigma,I_\tau} \le C\big(\|\phi\|_{L^2_{t_0}} + \|\bar{g}\|_{L^2_{t_0}} + \|\bar{g}\|_{L^\sigma_\lambda(\bar{I}_\tau)}\big).$$

This readily gives the desired $W^{2,\sigma}$ estimate on ψ in the intermediate region, which combined with (3.76) yields to (3.65), finishing the proof of the lemma. □

We recall next the weighted L^∞ norm and our global norm given in Definitions 2.5 and 2.6 respectively. It is clear that

$$\|\psi\|^{\nu}_{L^\infty_{t_0}} \le C\|\psi\|^{\nu}_{\infty,t_0}$$

since $z^{-1} \ge c > 0$ for a universal constant c.

The following L^∞ estimate follows as a consequence of estimates (3.20) and (3.64). To derive it, as we will see below we need to take $\sigma > n+1$, so let us define from now on $\sigma := n+2$. We have

Corollary 3.3. *Under the assumptions of Proposition* 3.3, *if* $\sigma = n+2$, *then the solution* ψ *satisfies the estimate*

$$\|\psi\|^{\nu}_{\infty,t_0} \le C\big(\|f\|^{\nu}_{L^2_{t_0}} + \|f\|^{\nu}_{\sigma,t_0}\big). \tag{3.83}$$

Proof. The bound on the norm $\|\psi\|^{\nu}_{L^\infty_{t_0}}$ readily follows from estimate (3.20) and Sobolev embedding. For the bound on $\|z^{-1}\psi\chi_{\{|x|\ge\xi(t)\}}\|^{\nu}_{L^\infty_{t_0}}$, by symmetry we may restrict ourselves to the region $\{-\infty < x < -\xi(t) + \beta^{-1}\ln 2\}$. Set

$$\phi(x,t) := \psi(x-\xi_0(t),t), \quad -\infty < x < \beta^{-1}\ln 2.$$

As in the proof of Lemma 3.4, it follows that ϕ satisfies equation (3.70) with $\bar g := g(x-\xi_0(t),t)$ and $g := f - C(\psi)$. Hence, $\tilde\phi(r,t)$ given by (3.72) satisfies equation (3.73) which is now strictly parabolic in the region of consideration $0 \le r < 2$, $t \le t_0$. Let $Q = B_{e^\epsilon} \times [\tau-1,\tau]$ and $\bar Q = B_2 \times [\tau-2,\tau]$, $\tau \le t_0$ and $e^\epsilon < 2$. Standard parabolic estimates imply the bound

$$\|\tilde\phi\|_{L^\infty(Q)} \le C\big(\|\tilde\phi\|_{L^\sigma(\bar Q)} + \|\tilde g\|_{L^\sigma(\bar Q)}\big)$$

since $\sigma > n+1$. Expressing everything in the original variables, using that

$$|\xi(t)-\xi_0(t)| = |h(t)| \le C|t|^{-\mu} < \epsilon$$

for $|t|$ sufficiently large, we conclude

$$\|z^{-1}\psi\chi_{\{|x|\ge\xi(t)\}}\|^{\nu}_{L^\infty_{t_0}} \le \|z^{-1}\psi\chi_{\{|x|\ge\xi_0(t)-\epsilon\}}\|^{\nu}_{L^\infty_{t_0}} \le \big(\|\psi\|^{\nu}_{\sigma,t_0} + \|g\|^{\nu}_{\sigma,t_0}\big).$$

Since $g = f - C(\psi)$, where $C(\psi)$ satisfies the bound (3.69), the desired bound readily follows from (3.64). □

We will finally summarize the results in this section in one result. This will play a crucial role in the construction of the solution of our nonlinear problem. We have shown the following result.

Proposition 3.4. *Let* $\mu, \nu \in [0,1)$ *and* $\sigma = n+2$, $n \ge 3$, *be fixed constants. Then, there is a number* $t_0 < 0$ *such that for any even function* f *on* $\mathbb{R}\times(-\infty,t_0]$ *with* $\|f\|^{\nu}_{*,\sigma} < \infty$, *satisfying the orthogonality conditions* (3.5)–(3.6) *and a function* h *on* $(-\infty,t_0]$ *with* $\|h\|^{\mu,1+\mu}_{1,\sigma,t_0} < \infty$ *there exists an ancient solution* $\psi = T(f)$ *of* (3.9) *on* $-\infty \le t \le t_0$ *also satisfying the orthogonality conditions* (3.5)–(3.6), *and the estimate*

$$\|\psi\|^{\nu}_{*,2,\sigma,t_0} \le C\|f\|^{\nu}_{*,\sigma,t_0}. \tag{3.84}$$

The constant C *depends only on dimension* n, ν *and* μ.

4. The nonlinear problem

Let X be the Banach space defined as in Definition 2.8 and let T and A be the operators as introduced in Section 2.3. In addition, for a given $\mu \in (0,1)$ and $t_0 < 0$, we set

$$K := \{(h,\eta) : (-\infty, t_0] \to \mathbb{R} : \|h\|_{1,\sigma,t_0}^{\mu,1+\mu} \le 1 \text{ and } \|\eta\|_{1,\sigma,t_0}^{1} \le C_0\} \tag{4.1}$$

where C_0 is the universal constant given by (5.16). We say that $h \in K$ or $\eta \in K$ if $(h,0) \in K$ or $(0,\eta) \in K$ respectively. Moreover, define

$$\Lambda := \{\psi \in X : \|\psi\|_{*,2,\sigma,\nu} \le 1\}. \tag{4.2}$$

Remark 4.1. If $(h,\eta) \in K$, then

$$\|h\|_{L^\infty(-\infty,t_0])} \le |t_0|^{-\mu} \quad \text{and} \quad \|\eta\|_{L^\infty(-\infty,t_0])} \le C_0|t_0|^{-1}. \tag{4.3}$$

In particular, by choosing $|t_0|$ sufficiently large we may assume that both h and η have small L^∞ norms. In addition,

$$\xi(t) := \frac{1}{2}\log(2b|t|) + h(t) = \frac{1}{2}\log(2b|t|) + o(1), \quad \text{as } |t| \to -\infty. \tag{4.4}$$

The main goal in this section is to prove the following proposition.

Proposition 4.1. *Let $\sigma = n+2$. There exist numbers $\nu \in (\frac{1}{2}, 1)$ and $t_0 < 0$, depending only on dimension n, such that for any given pair of functions (h,η) in K, there is a solution $\psi = \Psi(h,\eta)$ of* (2.14) *which satisfies the orthogonality conditions* (2.9)–(2.10). *Moreover, the following estimates hold:*

$$\|\Psi(h^1,\eta) - \Psi(h^2,\eta)\|_{*,2,\sigma,t_0}^{\nu} \le C|t_0|^{-\mu}\|h^1 - h^2\|_{1,\sigma,t_0}^{\mu,\mu+1} \tag{4.5}$$

and

$$\|\Psi(h,\eta^1) - \Psi(h,\eta^2)\|_{*,2,\sigma,t_0}^{\nu} \le C|t_0|^{-1+\nu}\|\eta^1 - \eta^2\|_{1,\sigma,t_0}^{1} \tag{4.6}$$

for any $(h^i,\eta) \in K$ and $(h,\eta^i) \in K$, $i = 1,2$, and $\mu < \min\{2\nu - 1, \gamma\}$, where $\gamma \in (0,1)$ is a positive number determined by Lemma 5.1 *and C is a universal constant.*

We will find a solution of (2.14) by the contraction mapping principle. To this end, we need suitable estimates on the operator $E(\psi)$. They are given in the following subsection, after which we will proceed with the proof of Proposition 4.1.

4.1. The estimation of the error term. We will next estimate the error term $E(\psi)$ in the $\|\cdot\|_{*,\sigma,t_0}^{\nu}$ norm and also establish its Lipschitz property with respect to ψ as well as h and η. We will begin by estimating the error term M in (2.6).

Lemma 4.1. *Let $\sigma = n+2$. There exist numbers $\nu = \nu_0(n) \in (\frac{1}{2}, 1]$ and $t_0 < 0$, depending on dimension n, such that for any $\nu \in (\frac{1}{2}, \nu_0]$ and $\mu < \min\{2\nu - 1, \gamma\}$ and any $(h,\eta) \in K$ (where the set K is defined with respect to this particular μ), we have*

$$\|z^{1-p}M\|_{*,\sigma,t_0}^{\nu_0} \le C$$

for a universal constant C.

Proof. Throughout the proof, C will denote various universal constants. Since all the functions involved, including M, are even in x, it will be sufficient to restrict our computation to the region $x > 0$. Notice in that region

$$w_2(x,t) := w(x+\xi(t)) \le w(x-\xi(t)) =: w_1(x,t).$$

We write $M = M_1 + M_2$, where

$$M_1 = \tilde{z}^p - (1+\eta)^p(w_1^p + w_2^p), \quad M_2 = [(1+\eta)^p - (1+\eta)](w_1^p + w_2^p) - \partial_t \tilde{z}^p$$

and set

$$\bar{M}_1 = z^{1-p} M_1 \quad \text{and} \quad \bar{M}_2 = z^{1-p} M_2.$$

We have

$$\begin{aligned} 0 \le \bar{M}_1 &= z^{1-p}(1+\eta)^p([(w_1+w_2)^p - w_1^p] - w_2^p) \\ &\le p(1+\eta)^p w_2 \int_0^1 \frac{(w_1 + s w_2)^{p-1}}{z^{p-1}}\, ds \end{aligned}$$

hence, using that $w_2 \le w_1 \le z$, obtain the bound

$$|\bar{M}_1| \le C w_2.$$

For the moment take $\sigma \ge 2$ to be any constant and $q > 0$. By the last bound and the estimate $z \le 2w_1$ which holds on $x > 0$, we compute

$$\begin{aligned} \left(\int_0^{\xi(t)} \bar{M}_1^\sigma z^{q\sigma}\, dx\right)^{\frac{1}{\sigma}} &\le C\left(\int_0^{\xi(t)} w^\sigma(x+\xi(t)) w^{q\sigma}(x-\xi(t))\, dx\right)^{\frac{1}{\sigma}} \\ &\le C\left(\int_0^{\xi(t)} e^{-\sigma(x+\xi(t))} e^{q\sigma(x-\xi(t))}\, dx\right)^{\frac{1}{\sigma}} \\ &= C e^{-(1+q)\xi(t)} \left(\int_0^{\xi(t)} e^{(q-1)\sigma x}\, dx\right)^{\frac{1}{\sigma}}. \end{aligned}$$

Recalling (4.4) we then conclude that

$$\left(\int_0^{\xi(t)} \bar{M}_1^\sigma z^{q\sigma}\, dx\right)^{\frac{1}{\sigma}} \le C\alpha(|t|)$$

where $\alpha(|t|) = |t|^{-\frac{1+q}{2}}$ if $q < 1$, $\alpha(|t|) = |t|^{-1}$ if $q > 1$, and $\alpha(|t|) = (\ln|t|)^{\frac{1}{\sigma}} |t|^{-1}$ if $q = 1$. On the other hand, recalling that $\beta := \frac{2}{n-2} = \frac{p-1}{2}$, we have

$$\begin{aligned} \left(\int_{\xi(t)}^\infty \bar{M}_1^\sigma z^{n\beta-\sigma}\, dx\right)^{\frac{1}{\sigma}} &\le C\left(\int_{\xi(t)}^\infty w^\sigma(x+\xi(t)) w^{n\beta-\sigma}(x-\xi(t))\, dx\right)^{\frac{1}{\sigma}} \\ &\le C\left(\int_{\xi(t)}^\infty e^{-\sigma(x+\xi(t))} e^{-(n\beta-\sigma)(x-\xi(t))}\, dx\right)^{\frac{1}{\sigma}} \\ &= C e^{-2\xi(t)} \left(\int_{\xi(t)}^\infty e^{-(n\beta)(x-\xi(t))}\, dx\right)^{\frac{1}{\sigma}} \\ &\le C|t|^{-1}. \end{aligned}$$

First, we combine the above estimates when $q = \beta$ and $\sigma = 2$, to conclude that on the set $\Lambda_\tau = \mathbb{R} \times [\tau, \tau + 1]$, $\tau < t_0 - 1$, we have $\|\bar{M}_1\|_{L^2(\Lambda_\tau)} \le C|\tau|^{-\nu_0}$, with $\nu_0 := \nu_0(\beta) > \frac{1}{2}$ for all β. It follows that

$$\|\bar{M}_1\|^{\nu_0}_{L^2_{t_0}} \le C.$$

Also, if we combine the above estimates for $\sigma = n + 2$ and $q = 2\beta + \theta$, with θ a small universal positive constant as in the proof of Proposition 3.3, we obtain (recall the definitions (3.75) and (3.78)) that $\|\bar{M}_1\|_{\sigma,\Lambda_\tau} \le C|\tau|^{-\nu_0}$, where $\nu_0 = \nu_0(\beta) > \frac{1}{2}$. By choosing $t_0 < -1$, we obtain

$$\|\bar{M}_1\|^{\nu_0}_{\sigma,t_0} \le C.$$

We conclude that

$$\|\bar{M}_1\|^{\nu_0}_{*,\sigma,t_0} \le C.$$

We will now estimate the term involving $\bar{M}_2$. Since $w_2 \le w_1 \le z$, we have

$$|\bar{M}_2| \le C(p)\big(|\eta| w_1 + |\dot{\xi}(t)||w'(x + \xi(t)) - w'(x - \xi(t))| + |\dot{\eta}(t)| z\big)$$

hence, using that $|w'(x)| \le C w(x)$, we obtain

$$|\bar{M}_2| \le C(|\eta| + |\dot{\eta}| + |\dot{\xi}|) z.$$

It follows that for any $q > 0$, $\sigma \ge 2$ and $\tau < t_0 - 1$, we have

$$\left(\iint_{\Lambda_\tau} |\bar{M}_2|^\sigma [z^{q\sigma} \chi_{\{|x|<\xi(t)\}} + z^{n\beta-\sigma} \chi_{\{|x|\ge\xi(t)\}}]\, dx\, dt\right)^{\frac{1}{\sigma}} \le C\left(\int_\tau^{\tau+1} (|\eta|^\sigma + |\dot{\xi}|^\sigma + |\dot{\eta}|^\sigma)\, dt\right)^{\frac{1}{\sigma}}.$$

By Definition 2.7, the right-hand side of the last estimate is bounded by $C|\tau|^{-1}$ if we assume that $\|\eta\|^1_{\sigma,t_0} \le C_0$ and $\|h\|^{\mu,1+\mu}_{\sigma,t_0} \le 1$, with $\mu > 0$. Hence, arguing as before, we easily conclude the bound

$$\|\bar{M}_2\|^{\nu_0}_{*,\sigma,t_0} \le C.$$

This finishes the proof of the lemma. □

The following corollary follows immediately from Lemma 4.1 by choosing $\nu = \nu(n)$ to be any number in $(\frac{1}{2}, \nu_0)$ and $t_0 < 0$ so that $C|t_0|^{-(\nu_0-\nu)} < \frac{1}{2}$. Let also $\mu < \min\{2\nu - 1, \gamma\}$, where $\gamma \in (0, 1)$ is a positive number determined by Lemma 5.1.

Corollary 4.1. *Let $\sigma = n + 2$ and let $\nu = \nu(n) \in (\frac{1}{2}, 1)$, $t_0 < 0$, μ be as above. Then there exist uniform constants $t_0 < 0$ and $C > 0$, depending on dimension n, such that for any $(h, \eta) \in K$, we have*

$$\|z^{1-p} M\|^\nu_{*,\sigma,t_0} \le C|t_0|^{-(\nu_0-\nu)}.$$

For the remaining of the subsection we will fix the parameters σ, μ and ν as in Corollary 4.1. We will next establish an L^∞ bound on $\frac{\psi}{z}$ which will be used very frequently in the rest of the article.

Claim 4.1. *For any function ψ on $\mathbb{R} \times (-\infty, t_0]$, we have*

$$\left\| \frac{\psi}{z} \right\|_{L^\infty_{t_0}} \leq C |t_0|^{\frac{1}{2}-\nu} \|\psi\|^{\nu}_{*,2,\sigma,t_0} \tag{4.7}$$

for a universal constant C. Hence, under the assumption $\|\psi\|^{\nu}_{*,2,\sigma,t_0} \leq 1$,

$$\left\| \frac{\psi}{z} \right\|_{L^\infty_{t_0}} \leq C |t_0|^{\frac{1}{2}-\nu} \tag{4.8}$$

and it can be made sufficiently small by taking $|t_0|$ sufficiently large.

Proof. Recalling the definitions of our norms from Section 2.2, we have

$$\left\| \frac{\psi}{z} \chi_{\{|x| \geq \xi(t)\}} \right\|_{L^\infty_{t_0}} \leq C |t_0|^{-\nu} \|\psi\|^{\nu}_{*,2,\sigma,t_0}. \tag{4.9}$$

On the other hand for any $\tau \leq t_0 - 1$, on $\Lambda_\tau = \mathbb{R} \times [\tau, \tau + 1]$ we have

$$\left\| \frac{\psi}{z} \chi_{\{|x| \leq \xi(t)\}} \right\|_{L^\infty(\Lambda_\tau)} \leq |\tau|^{-\nu} \|\psi\|^{\nu}_{*,2,\sigma,t_0} \|z^{-1}\|_{L^\infty(\Lambda_\tau \cap \{|x| \leq \xi(t)\})}.$$

To estimate $\|z^{-1}\|_{L^\infty(\Lambda_\tau \cap \{|x| \leq \xi(t)\})}$, we observe that

$$\min_{\{|x| \leq \xi(t)\}} z(x,t) \geq \min_{\{0 < x \leq \xi(t)\}} w(x - \xi(t)) = w(\xi(t)) \geq C|t|^{-\frac{1}{2}}$$

since (7.2) holds. Hence, $\|z^{-1}\|_{L^\infty(\Lambda_\tau \cap \{|x| \leq \xi(t)\})} \leq C|\tau|^{\frac{1}{2}}$. It follows that

$$\|z^{-1}\psi\|_{L^\infty(\Lambda_\tau \cap \{|x| \leq \xi(t)\})} \leq C|\tau|^{\frac{1}{2}-\nu} \|\psi\|^{\nu}_{*,2,\sigma,t_0}$$

implying the bound

$$\left\| \frac{\psi}{z} \chi_{\{|x| \leq \xi(t)\}} \right\|_{L^\infty_{t_0}} \leq C |t_0|^{\frac{1}{2}-\nu} \|\psi\|^{\nu}_{*,2,\sigma,t_0}. \tag{4.10}$$

The claim now follows from estimates (4.9)–(4.10) and our assumption $\|\psi\|^{\nu}_{*,2,\sigma,t_0} \leq 1$. □

We will next estimate the norm of the term $(1 - \partial_t)N(\psi)$ in (2.6).

Lemma 4.2. *There exist uniform constants $t_0 < 0$ and $C > 0$, depending on dimension n, such that for any functions $(h, \eta) \in K$ and $\psi \in \Lambda$, we have*

$$\|z^{1-p}(1 - \partial_t)N(\psi)\|^{\nu}_{*,\sigma,t_0} \leq C|t_0|^{\frac{1}{2}-\nu} \|\psi\|^{\nu}_{*,2,\sigma,t_0}.$$

Proof. We write $N(\psi) = N_1(\psi) + N_2(\psi)$, where

$$N_1(\psi) = (\tilde{z} + \psi)^p - \tilde{z}^p - p\tilde{z}^{p-1}\psi, \quad N_2(\psi) = p\psi z^{p-1}[(1 + \eta)^{p-1} - 1].$$

To estimate $\|z^{1-p}(1 - \partial_t)N_1(\psi)\|^{\nu}_{*,\sigma,t_0}$, we begin by observing that

$$z^{1-p}N_1(\psi) = (1 + \eta)^{p-1}\tilde{z}\left[\left(1 + \frac{\psi}{\tilde{z}}\right)^p - 1 - p\frac{\psi}{\tilde{z}}\right] = p(1 + \eta)^{p-1}\psi A(\psi) \tag{4.11}$$

where

$$A(\psi) = \int_0^1 \left(\left(1 + s\frac{\psi}{\tilde{z}} \right)^{p-1} - 1 \right) ds.$$

By (4.8) we have

$$|A(\psi)| \le C(p)\frac{|\psi|}{\tilde{z}} \le C\frac{|\psi|}{z} \tag{4.12}$$

and therefore we conclude the bound

$$\|z^{1-p}N_1(\psi)\|^{\nu}_{*,\sigma,t_0} \le C\|\psi\|_{*,\sigma,t_0}\left\|\frac{\psi}{z}\right\|_{L^\infty_{t_0}} \le C|t_0|^{\frac{1}{2}-\nu}\|\psi\|_{*,2,\sigma,t_0}.$$

It remains to estimate $z^{1-p}\partial_t N_1(\psi)$. Differentiating (4.11) in t, we get

$$\partial_t N_1(\psi) = \underbrace{p\partial_t\tilde{z}^{p-1}\psi A(\psi)}_{=:N_{11}} + \underbrace{p\tilde{z}^{p-1}\partial_t\psi A(\psi)}_{=:N_{12}} + \underbrace{p\tilde{z}^{p-1}\psi\partial_t A(\psi)}_{=:N_{13}}. \tag{4.13}$$

Using (4.12) we obtain, similarly as before, the bounds

$$|z^{1-p}N_{11}| \le C(|\dot{\xi}| + |\dot{\eta}|)|\psi|\frac{|\psi|}{z}, \quad |z^{1-p}N_{12}| \le C|\partial_t\psi|\frac{|\psi|}{z}.$$

To estimate the term $|z^{1-p}N_{13}(\psi)|$, we first observe that since $\frac{|\psi|}{\tilde{z}} < \frac{1}{2}$ by (4.8) and $|t_0| \gg 1$, we have

$$\begin{aligned} |\partial_t A(\psi)| &\le C\frac{|\tilde{z}\partial_t\psi - \tilde{z}_t\psi|}{\tilde{z}^2}\int_0^1\left(1 + s\frac{\psi}{\tilde{z}}\right)^{p-2} s\,ds \\ &\le C\frac{|\partial_t\psi| + (|\dot{\xi}| + |\dot{\eta}|)|\psi|}{z}. \end{aligned} \tag{4.14}$$

Hence

$$|z^{1-p}N_{13}(\psi)| \le C\frac{|\psi|}{z}(|\partial_t\psi| + (|\dot{\xi}| + |\dot{\eta}|)|\psi|). \tag{4.15}$$

Combining the above estimates with (4.7) gives

$$\|z^{1-p}\partial_t N_1(\psi)\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{\frac{1}{2}-\nu}\|\psi\|^{\nu}_{*,2,\sigma,t_0}\big(\|\psi\|^{\nu}_{*,2,\sigma,t_0} + \|(|\dot{\xi}| + |\dot{\eta}|)\psi\|^{\nu}_{*,\sigma,t_0}\big).$$

However, a direct computation shows that

$$\|(|\dot{\xi}| + |\dot{\eta}|)\psi\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-1}\big(\|\dot{\xi}\|^1_{\sigma,t_0} + \|\dot{\eta}\|^1_{\sigma,t_0}\big)\|\psi\|^{\nu}_{*,2,\sigma,t_0} \tag{4.16}$$

where

$$\|\dot{\eta}\|^1_{\sigma,t_0} \le \|\eta\|^1_{1,\sigma,t_0} \le 1 \quad \text{and} \quad \|\dot{\xi}\|^1_{\sigma,t_0} \le \frac{1}{2} + \|\dot{h}\|^1_{\sigma,t_0} \le \frac{1}{2} + \|h\|^{\mu,1+\mu}_{1,\sigma,t_0} \le 2.$$

Hence,

$$\|z^{1-p}\partial_t N_1(\psi)\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{\frac{1}{2}-\nu}\|\psi\|^{\nu}_{*,2,\sigma,t_0}.$$

Since $\|\eta\|^1_{*,1,\sigma,t_0} \le C_0$, a computation along the lines of the previous estimate also shows that

$$\|z^{1-p}(1-\partial_t)N_2(\psi)\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-1}\|\psi\|^{\nu}_{*,2,\sigma,t_0}. \tag{4.17}$$

The proof of the lemma is now complete. □

Lemma 4.3. *There exist $t_0 < 0$ and $C > 0$, depending on dimension n, such that for any functions $(h,\eta) \in K$ and $\psi \in \Lambda$, we have*

$$\|\bar{E}(\psi)\|^{\nu}_{*,\sigma,t_0} \le C\big(|t_0|^{-(\nu_0-\nu)} + |t_0|^{\frac{1}{2}-\nu}\|\psi\|^{\nu}_{*,2,\sigma,t_0}\big). \tag{4.18}$$

Proof. Let $Q(\psi)$ be as in (2.6). The estimate of the error term $N(\psi)$ given in Lemma 4.2, the estimate of the correction term $C(\psi)$ given in Corollary 3.2, and the bound

$$\|z^{1-p}\psi\partial_t z^{p-1}\|^{\nu}_{*,\sigma,t_0} \le C\||\dot{\xi}|\psi\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-1}\|\psi\|^{\nu}_{*,2,\sigma,t_0}$$

(where we have used the L^∞ bound on ψ given by Lemma 3.2) yield

$$\|Q(\psi)\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{\frac{1}{2}-\nu}\|\psi\|^{\nu}_{*,2,\sigma,t_0}. \tag{4.19}$$

This combined with the estimate in Lemma 4.1 easily imply (4.18). □

We will next show the Lipschitz property of $E(\psi)$ with respect to ψ.

Lemma 4.4. *There exist $t_0 < 0$ and $C > 0$, depending on dimension n, such that for any functions $(h,\eta) \in K$ and $\psi^1, \psi^2 \in \Lambda$, we have*

$$\|E(\psi^1) - E(\psi^2)\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{\frac{1}{2}-\nu}\|\psi^1 - \psi^2\|^{\nu}_{*,2,\sigma,t_0}. \tag{4.20}$$

Proof. We begin by observing that the bound

$$\|C(\psi^1,t) - C(\psi^2,t)\|^{\nu}_{*,\sigma,t_0} \le \frac{C}{\sqrt{|t_0|}}\|\psi^1 - \psi^2\|_{*,2,\sigma,t_0}$$

follows similarly as the bound in Corollary 3.2.

All the other estimates are similar to those in Lemma 4.2, so we will omit most of the details. Using the notation in the proof of Lemma 4.2, let us look at the estimate of the term

$$|z^{1-p}(N_{13}(\psi^1) - N_{13}(\psi^2))| \le C\big(|\psi^1-\psi^2||\partial_t A(\psi^1)| + |\psi^2||\partial_t A(\psi^1) - \partial_t A(\psi^2)|\big).$$

By (4.14) we have

$$|\psi^1-\psi^2||\partial_t A(\psi^1)| \le C\frac{|\psi^1-\psi^2|}{z}\big(|\partial_t\psi^1| + (|\dot{\xi}| + |\dot{\eta}|)|\psi^1|\big)$$

hence, by (4.8) applied to $\psi^1 - \psi^2$, (4.16) and the assumed bounds on h, η and ψ, we conclude

$$\||\psi^1-\psi^2||\partial_t A(\psi^1)|\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{\frac{1}{2}-\nu}\|\psi^1-\psi^2\|^{\nu}_{*,2,\sigma,t_0}.$$

To estimate the last term, we set

$$I(\psi) = \int_0^1 (1 + s\tilde{z}^{-1}\psi)^{p-2}s\,ds$$

so that

$$|\psi^2||\partial_t A(\psi^1) - \partial_t A(\psi^2)| \le \Lambda_1 + \Lambda_2$$

where

$$\Lambda_1 = |\psi^2| \frac{|\tilde{z}\partial_t \psi^1 - \partial_t \tilde{z} \psi^1|}{\tilde{z}^2} |I(\psi^1) - I(\psi^2)|$$

and

$$\Lambda_2 = |\psi^2| \frac{|\tilde{z}\partial_t(\psi^1 - \psi^2) - \partial_t \tilde{z}(\psi^1 - \psi^2)|}{\tilde{z}^2} |I(\psi^2)|.$$

The bound

$$\|\Lambda_2\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{\frac{1}{2}-\nu} \|\psi^1 - \psi^2\|^{\nu}_{*,2,\sigma,t_0}$$

follows by similar arguments as before. For the other term we have

$$\|\Lambda_1\|^{\nu}_{*,\sigma,t_0} \le C \left\| \frac{\psi^2}{z} \right\|_{L^\infty_{t_0}} \| |\partial_t \psi^1| + (|\dot{\xi}| + |\dot{\eta}|)|\psi^1| \|^{\nu}_{*,\sigma,t_0} \|I(\psi^1) - I(\psi^2)\|_{L^\infty_{t_0}}$$

where

$$\|\tilde{z}^{-1}\psi^2\|_{L^\infty_{t_0}} \le C|t_0|^{\frac{1}{2}-\nu}$$

by (4.8) and

$$\| |\partial_t \psi^1| + (|\dot{\xi}| + |\dot{\eta}|)|\psi^1| \|^{\nu}_{*,\sigma,t_0} \le C$$

by (4.16) and the assumed bounds on ψ, h and η. On the other hand, applying (4.7) to $\psi^1 - \psi^2$ and using that $\frac{|\psi|}{\tilde{z}} < \frac{1}{2}$ (by (4.8) and $|t_0| \gg 1$), we obtain

$$\|I(\psi^1) - I(\psi^2)\|_{L^\infty_{t_0}} \le C \left\| \frac{\psi^1 - \psi^2}{\tilde{z}} \right\|_{L^\infty_{t_0}} \le C|t_0|^{\frac{1}{2}-\nu} \|\psi^1 - \psi^2\|_{*,2,\sigma,t_0}.$$

Combining the above gives us the bound

$$\|\Lambda_2\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{\frac{1}{2}-\nu} \|\psi^1 - \psi^2\|^{\nu}_{*,2,\sigma,t_0}.$$

All other bounds can be obtained similarly. □

We will now show the Lipschitz property of the error term M with respect to h and η.

Lemma 4.5. *There exist $t_0 < 0$ and $C > 0$, depending on dimension n, such that for any functions $h, h_i, \eta, \eta_i \in K$, $i = 1, 2$, we have*

$$\|(z^1)^{1-p}(M(h^1, \eta) - M(h^2, \eta))\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-\mu} \|h^1 - h^2\|^{\mu,\mu+1}_{1,\sigma,t_0} \tag{4.21}$$

and

$$\|z^{1-p}(M(h, \eta^1) - M(h, \eta^2))\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{\nu-1} \|\eta^1 - \eta^2\|^1_{1,\sigma,t_0}. \tag{4.22}$$

Proof. The estimates follow by direct (yet tedious) calculation, along the lines of the proof of Lemma 4.1. Set

$$z^i(x,t) := \underbrace{w(x + \xi_i(t))}_{=: w_2^i} + \underbrace{w(x - \xi_i(t))}_{=: w_1^i}.$$

For the reason of dealing with even functions we restrict ourselves to the region $x \geq 0$, where $w_2^i \leq w_1^i \leq z^i$. Using the notation of Lemma 4.1, we have

$$
\begin{aligned}
&|(z^1)^{1-p}[M_1(h^1,\eta) - M_1(h^2,\eta)]| \\
&\quad = (z^1)^{1-p}(1+\eta)^p\Big[w_2^1 \int_0^1 (w_1^1 + s w_2^1)^{p-1}\,ds - w_2^2 \int_0^1 (w_1^2 + s w_2^2)^{p-1}\,ds\Big] \\
&\qquad + (z^1)^{1-p}(1+\eta)^p[-(w_2^1)^p + (w_2^2)^p] \\
&\quad = (z^1)^{1-p}(1+\eta)^p\Big[(w_2^1 - w_2^2)\int_0^1 (w_1^1 + s w_2^1)^{p-1}\,ds + ((w_2^2)^p - (w_2^1)^p)\Big] \\
&\qquad + (z^1)^{1-p}(1+\eta)^p w_2^2 \int_0^1 ((w_1^1 + s w_2^1)^{p-1} - (w_1^2 + s w_2^2)^{p-1})\,ds.
\end{aligned}
$$

From the bound

$$\frac{|w_2^1 - w_2^2|}{w_2^2} \leq C|h_1(t) - h_2(t)|, \quad i = 1,2,$$

we conclude the estimate

$$|(z^1)^{1-p}[M_1(h^1,\eta) - M_1(h^2,\eta)]| \leq C w_2^2 |h_1 - h_2|.$$

Having the above estimate, a similar computation as in Lemma 4.1 implies

$$\|(z^1)^{1-p}(M_1(h^1,\eta) - M_1(h^2,\eta))\|_{*,\sigma,t_0}^{\nu} \leq C|t|^{-\mu}\|h^1 - h^2\|_{\infty,t_0}^{\mu}.$$

For the M_2 term we have

$$
\begin{aligned}
(z^1)^{1-p}|M_2(h^1,\eta) - M_2(h^2,\eta)| &\leq C|\eta|\big(((w_1^1)^p - (w_1^2)^p) + ((w_2^1)^p - (w_2^2)^p)\big) \\
&\leq C|\eta||h^1 - h^2| w(x - \xi_1)
\end{aligned}
$$

implying

$$\|(z^1)^{1-p}(M_2(h^1,\eta) - M_2(h^2,\eta))\|_{*,\sigma,t_0}^{\nu} \leq C|t|^{\nu-1-\mu}\|h^1 - h^2\|_{\infty,t_0}^{\mu}.$$

To conclude (4.21) we observe that $\nu < 1$ and

$$\|h^1 - h^2\|_{\infty,t_0}^{\mu} \leq \|h^1 - h^2\|_{1,\sigma,t_0}^{\mu,\mu+1}.$$

Finally,to show the Lipschitz property of M in η, we use the bounds

$$|(z^1)^{1-p}(M(h,\eta^1) - M(h,\eta^2))| \leq C|\eta^1 - \eta^2| w_1$$

and

$$|\eta_1 - \eta_2| \leq |t_0|^{-1}\|\eta^1 - \eta^2\|_{\infty,t_0}^{1} \leq |t_0|^{-1}\|\eta^1 - \eta^2\|_{1,\sigma,t_0}^{1},$$

implying that

$$\|(z^1)^{1-p}(M(h,\eta^1) - M(h,\eta^2))\|_{*,\sigma,t_0}^{\nu} \leq C|t|^{\nu-1}\|\eta^1 - \eta^2\|_{1,\sigma,t_0}^{1}. \qquad \square$$

We will show the Lipschitz property of the error term

$$\tilde{E}(\psi)(h,\eta) := (1 - \partial_t)N(\psi) - p\psi\partial_t z^{p-1} - z^{p-1}(c_1(t)z + c_2(t)\bar{z})$$

with respect to h and η.

Lemma 4.6. *There exist $t_0 < 0$ and $C > 0$, depending on dimension n, so that for any functions $\psi \in \Lambda$, $h, h_i, \eta, \eta_i \in K$, $i = 1, 2$, we have*

$$\|(z^1)^{1-p}(\tilde{E}(\psi)(h^1,\eta) - \tilde{E}(\psi)(h^2,\eta))\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-\mu}\|h^1 - h^2\|^{\mu,\mu+1}_{1,\sigma,t_0} \tag{4.23}$$

and

$$\|z^{1-p}(\tilde{E}(\psi)(h,\eta^1) - \tilde{E}(\psi)(h,\eta^2))\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-1}\|\eta^1 - \eta^2\|^{1}_{1,\sigma,t_0}. \tag{4.24}$$

Proof. The estimates again follow from a direct (yet tedious) calculation, along the lines of the proof of Lemma 4.2. For example, regarding the term N_{13}, as in the proof of Lemma 4.2, we have

$$\begin{aligned}
&|(z^1)^{1-p}N_{13}(\psi)(h^1,\eta) - (z^2)^{1-p}N_{13}(\psi)(h^2,\eta)| \\
&\quad\le C|\psi|\left|\int_0^1\left(\left(1+s\frac{\psi}{\tilde{z}^1}\right)^{p-2}\frac{(\psi_t\tilde{z}^1 - \psi(\tilde{z}^1)_t)}{(\tilde{z}^1)^2} - \left(1+s\frac{\psi}{\tilde{z}^2}\right)^{p-2}\frac{(\psi_t\tilde{z}^2 - \psi(\tilde{z}^2)_t)}{(\tilde{z}^2)^2}\right)s\,ds\right| \\
&\quad\le C|\psi|\frac{|\psi_t\tilde{z}^1 - \psi(\tilde{z}^1)_t|}{(\tilde{z}^1)^2}\int_0^1\left|\left(1+s\frac{\psi}{\tilde{z}^1}\right)^{p-2} - \left(1+s\frac{\psi}{\tilde{z}^2}\right)^{p-2}\right|s\,ds \\
&\qquad\quad + C|\psi|\left|\psi_t\left(\frac{1}{\tilde{z}^2} - \frac{1}{\tilde{z}^1}\right) - \psi\left(\frac{(\tilde{z}^2)_t}{(\tilde{z}^2)^2} - \frac{(\tilde{z}^1)_t}{(\tilde{z}^1)^2}\right)\right|\int_0^1\left(1+s\frac{\psi}{\tilde{z}^2}\right)^{p-2}s\,ds \\
&\quad\le C|\psi||h^1 - h^2|\left(\frac{|\psi_t|}{\tilde{z}^1} + \frac{|\psi|}{\tilde{z}^1}(|\dot{\xi}_1| + |\dot{\xi}_2| + |\dot{\eta}|)\right).
\end{aligned}$$

Now it easily follows (similarly as in the proof of Lemma 4.2) that for $\psi \in \Lambda$, we have

$$\begin{aligned}
&\|(z^1)^{1-p}N_{13}(\psi)(h^1,\eta) - (z^2)^{1-p}N_{13}(\psi)(h^2,\eta)\|^{\nu}_{*,\sigma,t_0} \\
&\qquad\le C|t_0|^{\frac{1}{2}-\nu-\mu}\|h^1 - h^2\|^{\mu}_{\infty,t_0}\|\psi\|^{\nu}_{*,\sigma,t_0} \\
&\qquad\le C|t_0|^{\frac{1}{2}-\nu-\mu}\|h^1 - h^2\|^{\mu,\mu+1}_{1,\sigma,t_0}.
\end{aligned}$$

All other terms in $E(\psi)$ can be estimated similarly and estimate (4.23) follows.

Let us now look at the contraction of $E(\psi)$ in η. For example, we have

$$\begin{aligned}
&|z^{1-p}(N_{13}(\psi)(h,\eta^1) - N_{13}(\psi)(h,\eta^2))| \\
&\quad\le C|\psi|\left|(1+\eta^1)^{p-1}\left(\frac{\psi_t}{(1+\eta^1)z} - \psi\frac{(1+\eta^1)\dot{z} + \dot{\eta}^1 z}{(1+\eta^1)^2z^2}\right)\int_0^1\left(1+s\frac{\psi}{(1+\eta^1)z}\right)^{p-2}s\,ds\right. \\
&\qquad\quad\left. - (1+\eta^2)^{p-1}\left(\frac{\psi_t}{(1+\eta^2)z} - \psi\frac{(1+\eta^2)\dot{z} + \dot{\eta}_2 z}{(1+\eta^2)^2z^2}\right)\int_0^1\left(1+s\frac{\psi}{(1+\eta^2)z}\right)^{p-2}s\,ds\right| \\
&\quad\le C|\psi||\eta^1 - \eta^2|\left(\frac{|\psi_t| + |\psi| + |\psi|(|\dot{\xi}| + |\dot{\eta}_1|)}{z}\right) + C\frac{|\psi|}{z}(|\dot{\eta}_1 - \dot{\eta}_2| + |\dot{\xi}||\eta^1 - \eta^2|).
\end{aligned}$$

This easily implies (as in the proof of Lemma 4.2) the bound

$$\|z^{1-p}(N_{13}(\psi)(h,\eta^1) - N_{13}(\psi)(h,\eta^2))\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-\frac{1}{2}-\nu}\|\eta^1 - \eta^2\|^{1}_{1,\sigma,t_0}\|\psi\|^{\nu}_{*,2,\sigma,t_0}.$$

Furthermore,

$$|z^{1-p}(N_{12}(\psi)(h,\eta_1) - N_{12}(h,\eta_2))| \le C|\psi_t||\eta^1 - \eta^2|\left(1 + \frac{|\psi|}{z}\right)$$

implying that

$$\|z^{1-p}(N_{12}(\psi)(h,\eta^1) - N_{12}(h,\eta^2))\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-1}\|\eta^1 - \eta_2\|^1_{1,\sigma,t_0}\|\psi\|^{\nu}_{*,2,\sigma,t_0}.$$

The other terms in $E(\psi)$ can be estimated similarly and estimate (4.24) follows by recalling that $\|\psi\|^{\nu}_{*,2,\sigma,t_0} \le 1$ for all $\psi \in \Lambda$. □

4.2. Proof of Proposition 4.1. In this subsection we give the proof of Proposition 4.1 which claims the existence of a solution to the auxiliary equation (2.14) with the desired properties. We fix $(h,\eta) \in K$ and we show that $A : X \cap \Lambda \to X \cap \Lambda$, given by (2.14) defines a contraction and then use the fixed point theorem.

(a) There exists a universal constant $t_0 < 0$ for which $A(X \cap \Lambda) \subset X \cap \Lambda$. Indeed, assume that $\psi \in X \cap \Lambda$. By Proposition 3.4 we have

$$\|A(\psi)\|^{\nu}_{*,2,\sigma,t_0} = \|T(\bar{E}(\psi)\|^{\nu}_{*,2,\sigma,t_0} \le C\|\bar{E}(\psi)\|^{\nu}_{*,\sigma,t_0}$$

given that $\bar{E}(\psi)$ satisfies the orthogonality conditions (2.9) and (2.10). In addition, it is easy to see that for the second term on the right-hand side in (2.12), we have

$$\|c_1(t)z + c_2(t)\bar{z}\|^{\nu}_{*,\sigma,t_0} \le C\|E(\psi)\|^{\nu}_{*,\sigma,t_0}.$$

Hence,

$$\|A(\psi)\|^{\nu}_{*,2,\sigma,t_0} = \|T(\bar{E}(\psi)\|^{\nu}_{*,2,\sigma,t_0} \le C\|E(\psi)\|^{\nu}_{*,\sigma,t_0}.$$

Combining the last estimate with (4.18) and the bound $\|\psi\|^{\nu}_{*2,\sigma,t_0} \le 1$, we get

$$\|A(\psi)\|^{\nu}_{*,2,\sigma,t_0} \le 1$$

if $|t_0|$ is chosen sufficiently large.

(b) There exists a universal constant $t_0 < 0$ for which $A : X \cap \Lambda \to X \cap \Lambda$ defines a contraction map. For any $\psi^1, \psi^2 \in X \cap \Lambda$, Proposition 3.4 implies the bound

$$\|A(\psi^1) - A(\psi^2)\|^{\nu}_{*,2,\sigma,t_0} = \|T(\bar{E}(\psi^1) - \bar{E}(\psi^2))\|^{\nu}_{*,2,\sigma,t_0} \le C\|\bar{E}(\psi^1) - \bar{E}(\psi^2)\|^{\nu}_{*,\sigma,t_0}.$$

Similarly as above, we have

$$\|\bar{E}(\psi^1) - \bar{E}(\psi^2)\|^{\nu}_{*,\sigma,t_0} \le C\|E(\psi^1) - E(\psi^2)\|^{\nu}_{*,\sigma,t_0}.$$

The last two estimates with (4.20) yield the contraction bound

$$\|A(\psi^1) - A(\psi^2)\|^{\nu}_{*,2,\sigma,t_0} \le q\|\psi^1 - \psi^2\|^{\nu}_{*,2,\sigma,t_0}$$

with $q < 1$, provided that $|t_0|$ is chosen sufficiently large.

The above discussion and the fixed point theorem readily imply the existence of a unique fixed point $\psi = \Psi(h,\eta) \in X \cap \Lambda$ of the map A.

We will now continue with the proofs of (4.5) and (4.6).

(c) There exists a $t_0 < 0$ such that for any $(h^1, \eta), (h^2, \eta) \in K$, (4.5) holds. Since η is fixed, we will omit denoting the dependence on η. For simplicity we set $\psi^1 = \Psi(h^1, \eta)$ and $\psi^2 = \Psi(h^2, \eta)$. The estimate will be obtained by applying estimate (3.20). However, because each ψ^i satisfies the orthogonality conditions (2.9) and (2.10) with

$$\xi(t) = \xi^i(t) := \frac{1}{2}\log(2b|t|) + h^i(t),$$

the difference $\psi^1 - \psi^2$ does not satisfy an exact orthogonality condition. To overcome this technical difficulty, we will consider instead the difference $Y := \psi^1 - \bar{\psi}^2$, where

$$\bar{\psi}^2(x,t) = \psi^2(x,t) - \lambda_1(t)w'(x - \xi^1(t)) - \lambda_2(t)w(x - \xi^1(t))$$

with

$$\lambda_1(t) = \int \psi^2(x - \xi^1(t), t)w'(x)w^{p-1}dx,$$
$$\lambda_2(t) = \int \psi^2(x - \xi^1(t), t)w(x)w^{p-1}dx.$$

Clearly, Y satisfies the orthogonality conditions (2.9) and (2.10) with $\xi(t) = \xi^1(t)$. Denote by L_t^1 the operator

$$L_t^1 Y := p(z^1)^{p-1}\partial_t Y - [\partial_{xx}Y - Y + p(z^1)^{p-1}Y].$$

Since each of the ψ^i satisfies equation (2.14), it follows that $Y := \psi^1 - \bar{\psi}^2$ satisfies

$$\begin{aligned} L_t^1 Y = {} & M(h^1) - M(h^2) + (z^1)^{p-1}(\hat{E}(\psi^1, h^1) - \hat{E}(\psi^2, h^2)) \\ & - L_t^1(\psi^2 - \bar{\psi}^2) + ((z^2)^{p-1} - (z^1)^{p-1})(1 - \partial_t)\psi^2 \end{aligned}$$

where for $i = 1, 2$, we denote by $M^i := M(h^i)$ and by

$$\hat{E}(\psi^i, h^i) := (z^1)^{1-p}\big[(1 - \partial_t)N(\psi^i) - p\psi^i\partial_t(z^i)^{p-1} - (z^i)^{p-1}(c_1^i(t)z^i + c_2^i(t)\bar{z}^i)\big]$$

with $M(h^i)$, $N(\psi^i)$ and c_1^i, c_2^i defined in (2.7), (2.8) and (2.12) respectively.

We next observe that estimate (3.20) holds for any even solution Y of equation

$$L_t^1 Y = (z^1)^{p-1} f$$

as long as the solution Y itself, and not necessarily f, satisfies the orthogonality conditions (2.9) and (2.10). Indeed, the a priori estimate

$$\|Y\|_{*,2,\sigma,t_0}^{\nu} \le C(\|Y\|_{*,\sigma,t_0}^{\nu} + \|f\|_{*,\sigma,t_0}^{\nu})$$

holds for any solution Y and the bound $\|Y\|_{*,\sigma,t_0}^{\nu} \le C\|f\|_{*,\sigma,t_0}^{\nu}$, based on the contradiction argument given in Proposition 3.2 can be shown to hold for any even solution Y that satisfies (2.9) and (2.10). Hence, we have

$$\|Y\|_{*,2,\sigma,t_0}^{\nu} \le C\|(z^1)^{1-p}L_t^1 Y\|_{*,\sigma,t_0}^{\nu}. \tag{4.25}$$

Claim 4.2. *We have*

$$\|(z^1)^{1-p}L_t^1 Y\|_{*,\sigma,t_0}^{\nu} \le C|t_0|^{\frac{1}{2}-\nu}\|Y\|_{*,2,\sigma,t_0}^{\nu} + C|t_0|^{-\mu}\|h^1 - h^2\|_{1,\sigma,t_0}^{\mu,\mu+1}.$$

Proof. By (4.21), we have

$$\|(z^1)^{1-p}(M(h^1)-M(h^2))\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-\mu}\|h^1-h^2\|^{\mu,\mu+1}_{1,\sigma,t_0}.$$

Also, by combining (4.20) and (4.23), we have

$$\|\hat{E}(\psi^1,h^1)-\hat{E}(\psi^2,h^2)\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{\frac{1}{2}-\nu}\|\psi^1-\psi^2\|^{\nu}_{*,2,\sigma,t_0} + C|t_0|^{-\mu}\|h^1-h^2\|^{\mu,\mu+1}_{1,\sigma,t_0}.$$

Also, since $\|(1-\partial_t)\psi^2\|^{\nu}_{*,\sigma,t_0} \le 2\|\psi^2\|^{\nu}_{*,2,\sigma,t_0} \le 2$,

$$\left\|\frac{(z^2)^{p-1}-(z^1)^{p-1}}{(z^1)^{p-1}}(1-\partial_t)\psi^2\right\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-\mu}\|h^1-h^2\|^{\mu,\mu+1}_{1,\sigma,t_0}.$$

For the term $L_t^1(\psi^2-\bar{\psi}^2)$ we observe that since both $w(x)$ and $w'(x)$ are eigenfunctions of the operator L_0 given in (3.3), and $w(x)$ and all its derivatives are bounded in $\mathbb{R}$, we have

$$(z^1)^{1-p}|L_t^1(\psi^2-\bar{\psi}^2)| \le C\sum_{i=1}^{2}(|\lambda_i|+|\dot{\lambda}_i|+|\lambda_i||\dot{\xi}^1|).$$

Let us now estimate $|\lambda_i(t)|$ and $|\dot{\lambda}_i(t)|$. Using the orthogonality condition (2.9) satisfied by ψ^2 (with $\xi=\xi^2$), we have

$$\begin{aligned}|\lambda_1(t)| &= \left|\int_{\mathbb{R}}(\psi^2(x-\xi^1)-\psi^2(x-\xi^2))w'(x)w^{p-1}\,dx\right| \\ &\le C|(h^1-h^2)(t)|\|\psi^2(\cdot,t)\|_{L^2}.\end{aligned}$$

Similarly, one can see that

$$|\dot{\lambda}_1(t)| \le C\big(|(h^1-h^2)(t)|\|\partial_t\psi^2(\cdot,t)\|_{L^2}+|(\dot{h}^1-\dot{h}^2)(t)|\|\psi^2(\cdot,t)\|_{L^2}\big).$$

The estimates for $|\lambda_2(t)|$ and $|\dot{\lambda}_2(t)|$ are the same. Combining the last estimates readily yields the bound

$$\|(z^1)^{1-p}|L_t^1(\psi^2-\bar{\psi}^2)\|^{\nu}_{*,\sigma,t_0} \le C|t_0|^{-\mu}\|h^1-h^2\|^{\mu,\mu+1}_{1,\sigma,t_0}.$$

To finish the proof of the claim, we need to show that

$$\|\psi^1-\psi^2\|^{\nu}_{*,2,\sigma,t_0} \le \|Y\|^{\nu}_{*,2,\sigma,t_0} + C|t_0|^{-\mu}\|h^1-h^2\|^{\mu,\mu+1}_{1,\sigma,t_0}.$$

Since $\|\psi^1-\psi^2\|^{\nu}_{*,2,\sigma,t_0} \le \|Y\|^{\nu}_{*,2,\sigma,t_0} + \|\lambda_1 w'(x-\xi^1)+\lambda_2 w(x-\xi^1)\|^{\nu}_{*,2,\sigma,t_0}$, this estimate readily follows from the previous bounds on λ_i. □

The proof of (4.5) now readily follows by combining (4.25) and the above claim and choosing $|t_0|$ sufficiently large.

(d) There exists $t_0<0$ such that for any $(h,\eta^1),(h,\eta^2)\in K$, (4.6) holds. This proof is an easy consequence of (4.22), (4.20), (4.24) and (3.20), since for $\psi^1=\Psi(h,\eta^1)$ and $\psi^2=\Psi(h,\eta^2)$, we have

$$\|\psi^1-\psi^2\|^{\nu}_{*,2,\sigma,t_0} = \|T(\bar{E}(\psi^1,\eta^1))-T(\bar{E}(\psi^2,\eta^2))\|^{\nu}_{*,2,\sigma,t_0}$$

where now the operator T depends only on h (not the η_i) and h is fixed.

5. Solving for ξ and η

We recall the definition of K given by (4.1). In the previous section we have established that for any given $(h,\eta)\in K$, there exists a solution $\psi=\Psi(h,\eta)$ of the auxiliary equation (2.14). Recall that $c_1(t)$ and $c_2(t)$ are chosen so that the error term

$$\bar{E}(\psi) := E(\psi) - (c_1(t)z + c_2(t)\bar{z})$$

satisfies the orthogonality conditions (2.9) and (2.10) whenever ψ does. The error term $E(\psi)$ is given by (2.6). Thus $\psi=\Psi(h,\eta)$ defines a solution to our original equation (2.5) if we manage to adjust the parameter functions (h,η) in such a way that $c_1\equiv 0$ and $c_2\equiv 0$. This is equivalent to choosing (h,η) so that

$$\int_{-\infty}^{\infty} E(\psi)w^p(x+\xi(t))\,dx = 0, \tag{5.1}$$

$$\int_{-\infty}^{\infty} E(\psi)w'(x+\xi(t))w^{p-1}(x+\xi(t))\,dx = 0. \tag{5.2}$$

In fact, the main result in this section is the following.

Proposition 5.1. *There exists $(h,\eta)\in K$ such that* (5.1)–(5.2) *are satisfied. It follows that the solution $\psi=\Psi(h,\eta)$ of the auxiliary equation* (2.14) *given by Proposition* 4.1 *defines a solution of our original equation* (2.5) *and Theorem* 1.1 *holds.*

5.1. Computation of error projections. The proof of Proposition 5.1 is based on careful expansions for the projections of the error terms (given by the left-hand sides of (5.1) and (5.2)) which lead us to a system of ODE for the functions $\xi := \xi(t)$ and $\eta := \eta(t)$. We will then solve this system by employing the fixed point theorem. We will see that the main order terms in the system are all coming from the projections of the term $z^{1-p}M$ in (2.6). Let us first expand these projections in terms of ξ, η and their derivatives.

Lemma 5.1 (Projections of the error term M). *We have*

$$\int_{-\infty}^{\infty} Mw^p(x+\xi)z^{1-p}\,dx = c_1\Big(\dot\eta - \frac{p-1}{p}\eta - ae^{-2\xi}\Big) + \mathcal{R}_1(\xi,\dot\xi,\eta,\dot\eta)$$

and

$$\int_{-\infty}^{\infty} Mw'(x+\xi)w^{p-1}(x+\xi)z^{1-p}dx = c_2(\dot\xi + be^{-2\xi}) + \mathcal{R}_2(\xi,\dot\xi,\eta,\dot\eta)$$

where c_1, c_2 are universal constants and

$$a = \frac{(p-1)\int_0^\infty w^p e^x\,dx + p\int_{-\infty}^0 w^p e^x\,dx}{p\int w^{p+1}\,dx} \quad \text{and} \quad b = \frac{\int_0^\infty w^p e^{-x}\,dx}{p\int w'^2 w^{p-1}\,dx}. \tag{5.3}$$

Moreover,

$$\|\mathcal{R}_i(\xi,\dot\xi,\eta,\dot\eta)\|_{\sigma,t_0}^{1+\gamma} \le C \quad \text{for } i\in\{1,2\},$$

and for some $0<\gamma<1$ that depends on dimension only.

Proof. We will use the notation of previous sections. Let us write $M = M_1+M_2+M_3$ with

$$M_1 = (1+\eta)^p((w_1+w_2)^p - w_2^p), \quad M_2 = -(1+\eta)^p w_1^p$$

and

$$M_3 = \underbrace{((1+\eta)^p - (1+\eta))(w_1^p + w_2^p)}_{=:M_{31}} - \underbrace{p(1+\eta)^{p-1}\dot{\eta} z^p}_{=:M_{32}} - \underbrace{p(1+\eta)^p z^{p-1}\dot{\xi}(\partial_x w_2 - \partial_x w_1)}_{=:M_{33}}.$$

We first compute the expansion of the term $\int z^{1-p} M w_2^p \, dx$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} M_1 \frac{w_2^p}{z^{p-1}} \, dx &= p(1+\eta)^p \int_{-\infty}^{\infty} \left[w_1 \int_0^1 (w_2 + s w_1)^{p-1} \, ds \right] \frac{w_2^p}{z^{p-1}} \, dx \\
&= p(1+\eta)^p \Big(\int_{-\infty}^{\infty} w_2^{p-1} w_1 \frac{w_2^p}{z^{p-1}} \, dx \\
&\qquad + (p-1) \int_{-\infty}^{\infty} [w_1^2 \int_0^1 (w_2 + s w_1)^{p-2} (1-s) ds] \frac{w_2^p}{z^{p-1}} \, dx \Big)
\end{aligned}$$

where we have used the notation $\bar{w}(x) := w(x) + w(x - 2\xi)$. We next analyze the terms on the right-hand side of the last equation. For the first term we have

$$\begin{aligned}
\int_{-\infty}^{\infty} w_2^{p-1} w_1 \frac{w_2^p}{z^{p-1}} \, dx &= \int_{-\infty}^{\infty} w^{p-1} w(x - 2\xi) \frac{w^p}{\bar{w}^{p-1}} \, dx \\
&= \int_{x \le 2\xi} w(x - 2\xi) w^p \, dx + g_1(2\xi) \\
&= e^{-2\xi} \int_{x \le 2\xi} e^x w^p \, dx + g_1(2\xi) \\
&= e^{-2\xi} \int_{-\infty}^{\infty} e^x w^p \, dx + g_1(2\xi)
\end{aligned}$$

where we denote by $g_1(2\xi)$ various error terms having the following decay:

$$|g_1(2\xi)| \le C e^{-(1+\gamma)2\xi}, \quad \text{with } 0 < \gamma < 1.$$

The other term turns out to be of a lower order and absorbable in $g_1(2\xi)$. To see that, since $w_2 \le w_1$ for $x > 0$ and $w_1 \le w_2$ for $x < 0$, we have

$$\begin{aligned}
\int w_1^2 \frac{w_2^p}{z^{p-1}} \left[\int_0^1 (w_2 + s w_1)^{p-2} (1-s) \, ds \right] dx &\le C \int_{x \ge 0} w_1 w_2^p \, dx + C \int_{x \le 0} w_1^2 w_2^{p-1} \, dx \\
&\le g_1(2\xi).
\end{aligned}$$

For the term M_2 we have

$$\begin{aligned}
-\int_{-\infty}^{\infty} M_2 \frac{w_2^p}{z^{p-1}} \, dx &= (1+\eta)^p \int_{-\infty}^{\infty} \frac{w^p(x - 2\xi)}{\bar{w}^{p-1}} w^p \, dx \\
&= \int_0^{2\xi} w^p(x - 2\xi) w \, dx + g_2(2\xi, \eta) \\
&= \int_{-2\xi}^0 w^p w(y + 2\xi) \, dy + g_2(2\xi, \eta) \\
&= e^{-2\xi} \int_{-\infty}^0 w^p e^{-y} \, dy + g_2(2\xi, \eta) \\
&= e^{-2\xi} \int_0^{\infty} w^p e^x \, dx + g_2(2\xi, \eta)
\end{aligned}$$

where we denote by $g_2(2\xi,\eta)$ various error terms having the following behavior:

$$|g_2(2\xi,\eta)| \le Ce^{-2\xi}(e^{-2\xi\gamma}+|\eta|), \quad \text{with } \gamma>0.$$

Furthermore,

$$\begin{aligned}\int M_{31}\frac{w_2^p}{z^{p-1}}\,dx &= [(1+\eta)^p-(1+\eta)]\bigg(\int_{-\infty}^{\infty} w^{2p}\bar{w}^{1-p}\,dx \\ &\qquad + \int_{-\infty}^{\infty} w^p w^p(x-2\xi)\bar{w}^{1-p}\,dx\bigg) \\ &= (p-1)\eta\int_{-\infty}^{\infty} w^{2p}\bar{w}^{1-p}\,dx + g_3(2\xi,\eta) \\ &= (p-1)\eta\int_{-\infty}^{\infty} w^{p+1}\,dx + g_3(2\xi,\eta)\end{aligned}$$

where we denote by $g_3(2\xi,\eta)$ various error terms with the behavior

$$|g_3(2\xi,\eta)| \le C\big(e^{-2\xi}(e^{-2\xi\gamma}+|\eta|)+|\eta|^2\big), \quad \text{with } \gamma>0.$$

Also,

$$\begin{aligned}\int_{-\infty}^{\infty} M_{32}\frac{w_2^p}{z^{p-1}}\,dx &= p(1+\eta)^{p-1}\dot{\eta}\int_{-\infty}^{\infty} z w_2^p\,dx \\ &= p\dot{\eta}\int_{-\infty}^{\infty} w^{p+1}\,dx + g_4(2\xi,\dot{\eta})\end{aligned}$$

with $|g_4(2\xi,\dot{\eta})| \le Ce^{-2\xi}|\dot{\eta}|$, and

$$\begin{aligned}\int_{-\infty}^{\infty} M_{33}\frac{w_2^p}{z^{p-1}}\,dx &= p(1+\eta)^p\dot{\xi}\int_{-\infty}^{\infty}(\partial_x w_2-\partial_x w_1)w_2^p\,dx \\ &= p\dot{\xi}\int_{-\infty}^{\infty} w'(x-2\xi)w^p\,dx + g_5(2\xi,\dot{\xi},\eta) \\ &= g_5(2\xi,\dot{\xi},\eta)\end{aligned}$$

with

$$|g_5(2\xi,\dot{\xi},\eta)| \le C(e^{-2\xi}(|\eta|+|\dot{\xi}|)+|\dot{\xi}||\eta|).$$

Combining the previous estimates for M_1, M_2, M_3, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} M\frac{w_2^p}{z^{p-1}}\,dx &= -p\Big(\dot{\eta}-\frac{p-1}{p}\eta\Big)\int_{-\infty}^{\infty} w^{p+1}\,dx \\ &\quad + e^{-2\xi}\Big((p-1)\int_0^{\infty} e^x w^p\,dx + p\int_{-\infty}^0 e^x w^p\,dx\Big) \\ &\quad + \mathcal{R}_1(\xi,\dot{\xi},\eta,\dot{\eta})\end{aligned} \tag{5.4}$$

where

$$|\mathcal{R}_1(\xi,\dot{\xi},\eta,\dot{\eta})| \le C(e^{-2\xi}(e^{-2\gamma\xi}+|\eta|+|\dot{\xi}|+|\dot{\eta}|)+|\eta|(|\dot{\xi}|+|\eta|+|\dot{\eta}|)). \tag{5.5}$$

Let us now expand $\int M z^{1-p}\partial_x w_2 w_2^{p-1}\,dx$. Similarly as before, analyzing term by term we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} M_1\partial_x w_2\frac{w_2^{p-1}}{z^{p-1}}\,dx &= p(1+\eta)^p\int_{-\infty}^{\infty} w_2^{p-1}w_1\partial_x w_2\frac{w_2^{p-1}}{z^{p-1}}\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\\ &= p\int_{x\le 2\xi} w^{p-1}w'w(x-2\xi)\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\\ &= pe^{-2\xi}\int_{-\infty}^{\infty} w^{p-1}w'e^x\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\\ &= -e^{-2\xi}\int_{-\infty}^{\infty} w^p e^x\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\end{aligned}$$

where by $\mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})$ we have denoted various error terms that have the same behavior as in (5.5). Also,

$$\begin{aligned}-\int_{-\infty}^{\infty} M_2\partial_x w_2\frac{w_2^{p-1}}{z^{p-1}}\,dx &= (1+\eta)^p\int \frac{w^p(x-2\xi)}{\bar{w}^{p-1}(x)}w'w^{p-1}\,dx\\ &= \int_0^{2\xi} w^p(x-2\xi)w'\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\\ &= \int_{-2\xi}^{0} w^p w'(y+2\xi)\,dy + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\\ &= -e^{-2\xi}\int_0^{\infty} w^p e^x\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\end{aligned}$$

where $\mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})$ is the error term satisfying (5.5). Next, using that $\int w'w^p\,dx = 0$ and that

$$1-\left(\frac{w}{\bar{w}}\right)^{p-1} \le \begin{cases} C(p)(1-\frac{w}{\bar{w}}) & \text{for } x<\xi,\\ 1 & \text{otherwise,}\end{cases}$$

we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} M_{31}\partial_x w_2\frac{w_2^{p-1}}{z^{p-1}}\,dx &= (p-1)\eta\int_{-\infty}^{\infty} w^p w'\frac{w^{p-1}}{\bar{w}^{p-1}}\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\\ &= (p-1)\eta\int_{-\infty}^{\infty}\left(\left(\frac{w}{\bar{w}}\right)^{p-1}-1\right)w'w^p\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\\ &= \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})\end{aligned}$$

where $\mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta})$ satisfies (5.5). Using again that $\int w'w^p\,dx = 0$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} M_{32}\partial_x w_2\frac{w_2^{p-1}}{z^{p-1}}\,dx &= p(1+\eta)^{p-1}\dot{\eta}\int_{-\infty}^{\infty}\bar{w}w'w^{p-1}\,dx\\ &= p\dot{\eta}\int_{-\infty}^{\infty} w(x-2\xi)w'w^{p-1}\,dx\\ &= \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta}).\end{aligned}$$

Finally,

$$\begin{aligned}\int M_{33}\partial_x w_2\frac{w_2^{p-1}}{z^{p-1}}\,dx &= p(1+\eta)^{p-1}\dot{\xi}\int (w'(x)-w'(x-2\xi))w'w^{p-1}\,dx\\ &= p\dot{\xi}\int_{-\infty}^{\infty}(w')^2w^{p-1}\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta}).\end{aligned}$$

Combining the above estimates, we conclude the bound

$$\int_{-\infty}^{\infty} M\partial_x w_2 \frac{w_2^{p-1}}{z^{p-1}}\,dx = -p\dot{\xi}\int_{-\infty}^{\infty}(w')^2 w^{p-1}\,dx - e^{-2\xi}\int_0^{\infty} w^p e^{-x}\,dx + \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta}). \tag{5.6}$$

Combining (5.4), (5.6) and (5.5) finishes the proof of the lemma. □

As an immediate corollary of the previous lemma we obtain:

Corollary 5.1. *Set $Q(\psi) := E(\psi) - Mz^{1-p}$. With the same notation as in Lemma* 5.1*, equations* (5.1) *and* (5.2) *are equivalent to the system*

$$\dot{\eta} - \frac{p-1}{p}\eta - ae^{-2\xi} = \mathcal{R}_1(\xi,\dot{\xi},\eta,\dot{\eta}) + G_1(\psi,\xi,\eta) \tag{5.7}$$

and

$$\dot{\xi} + be^{-2\xi} = \mathcal{R}_2(\xi,\dot{\xi},\eta,\dot{\eta}) + G_2(\psi,\xi,\eta) \tag{5.8}$$

where

$$G_1(\psi,\xi,\eta) := c_1^{-1}\int_{-\infty}^{\infty} Q(\psi)w^p(x+\xi)\,dx$$

and

$$G_2(\psi,\xi,\eta) := c_2^{-1}\int_{-\infty}^{\infty} Q(\psi)w'(x+\xi)w^{p-1}(x+\xi)\,dx.$$

The error terms $\mathcal{R}_i(\xi,\dot{\xi},\eta,\dot{\eta})$ satisfy

$$\|\mathcal{R}_i(\xi,\dot{\xi},\eta,\dot{\eta})\|_{\sigma,t_0}^{1+\gamma} \le C \quad \textit{for } i \in \{1,2\}.$$

Remark 5.1. If we look at the proof of Lemma 5.1, we can trace all the error terms we have denoted by $\mathcal{R}_i(\xi,\dot{\xi},\eta,\dot{\eta})$. Observe that

$$\begin{aligned} |\partial_\eta \mathcal{R}_i| + |\partial_\xi \mathcal{R}_i| &\le Ce^{-2\xi}(e^{-2\gamma\xi} + |\dot{\xi}| + |\eta| + |\dot{\eta}| + 1),\\ |\partial_{\dot{\eta}} \mathcal{R}_i| + |\partial_{\dot{\xi}} \mathcal{R}_i| &\le C(e^{-2\xi} + |\eta|). \end{aligned} \tag{5.9}$$

Our strategy in solving system (5.7)–(5.8) is as follows: For given $\xi := \frac{1}{2}\log(2b|t|) + h$ with $\|h\|_{1,\sigma,t_0}^{\mu,\mu+1} \le 1$, we will first find a solution $\eta(\xi)$ to (5.7) by the fixed point theorem. The existence of (ξ,η) will be given by plugging $\eta(\xi)$ in (5.8) and applying the fixed point theorem once more.

5.2. Solving for η. In this subsection we will fix a function h on $(-\infty, t_0]$ with

$$\|h\|_{1,\sigma,t_0}^{\mu,\mu+1} \le 1$$

and solve the equation

$$\dot{\eta} - \frac{p-1}{p}\eta = F_h(\eta,\dot{\eta},\psi) \tag{5.10}$$

where

$$F_h(\eta,\dot\eta,\psi) := ae^{-2\xi} + \mathcal{R}_1(\xi,\dot\xi,\eta,\dot\eta) + G_1(\psi,h,\eta) \tag{5.11}$$

and $\mathcal{R}_1(\xi,\dot\xi,\eta,\dot\eta)$, $G_1(\psi,h,\eta)$ are as in Corollary 5.1. Recall that for any given $(h,\eta)\in K$, $\psi = \Psi(h,\eta)$ is the solution of (2.14), which was proved in Proposition 4.1. For simplicity we will denote, most of the time, those functions as F_h, R_1 and G_1 respectively.

Let $\lambda = \frac{p-1}{p}$. A function η is a solution of equation (5.10) on $(-\infty,t_0]$ if

$$A(\eta)(t) := -\int_t^{t_0} e^{\lambda(t-s)} F_h(\eta(s),\dot\eta(s),\psi(s))\,ds \tag{5.12}$$

satisfies $A(\eta)=\eta$. We have the following result.

Proposition 5.2. *For any fixed $h\in K$ there is an $\eta=\eta(h)\in K$ so that $A(\eta)=\eta$. Moreover, for any $h_1,h_2\in K$, we have*

$$\|\eta(h^1)-\eta(h^2)\|^1_{1,\sigma,t_0} \le C|t_0|^{-\delta}\|h^1-h^2\|^{\mu,\mu+1}_{1,\sigma,t_0} \tag{5.13}$$

where $t_0<0$ and C are universal constants and $\delta>0$ is a small constant depending on μ and ν.

Proof. Let A be the operator defined by (5.12).

(a) There exist a universal constant $t_0<0$ so that K is invariant under A, namely $A(K)\subset K$. We will first show that for $\sigma=n+2$ and $|t_0|$ sufficiently large, we have

$$\sup_{\tau\le t_0}|\tau||A(\eta)| \le C\|F_h\|^1_{2,t_0} \le C\|F_h\|^1_{\sigma,t_0}. \tag{5.14}$$

Indeed, if $t_0<-1$, then for $\tau<t_0$,

$$\begin{aligned}
|\tau||A(\eta)| &\le Ce^{\lambda\tau}|\tau|\sum_{j=0}^{[t_0-\tau]}\int_{\tau+j}^{\tau+j+1} e^{-\lambda s}|F_h(s)|\,ds\\
&\le C\|F_h\|^1_{2,t_0}e^{\lambda\tau}|\tau|\sum_{j=0}^{[t_0-\tau]}\frac{e^{-\lambda(\tau+j)}}{|\tau+j|}\\
&\le C\|F_h\|^1_{2,t_0}e^{\lambda\tau}|\tau|\int_\tau^{t_0}\frac{e^{-\lambda s}}{|s|}\,ds\\
&\le C\|F_h\|^1_{2,t_0}
\end{aligned}$$

since

$$e^{\lambda\tau}|\tau|\int_t^{t_0}\frac{e^{-\lambda s}}{|s|}\,ds \le C$$

for a uniform constant C. Denoting for simplicity by $I_\tau=[\tau,\tau+1]$ and using (5.14), observe that

$$\begin{aligned}
\|A(\eta)\|^1_{1,\sigma,t_0} &= \sup_{\tau\le t_0-1}|\tau|\|A(\eta)\|_{L^\sigma(I_\tau)} + \sup_{\tau\le t_0-1}|\tau|\left\|\frac{d}{dt}A(\eta)\right\|_{L^\sigma(I_\tau)}\\
&\le C\|F_h\|^1_{\sigma,t_0}
\end{aligned} \tag{5.15}$$

where C is a uniform constant. Next we want to use the previous estimate to show that $A(K) \subset K$, for an appropriately chosen constant C_0. By (5.11) we have

$$\begin{aligned}\|F_h\|^1_{\sigma,t_0} &:= \sup_{\tau \le t_0 - 1} |\tau| \|F_h\|_{L^\sigma(I_\tau)} \\ &\le \sup_{\tau \le t_0 - 1} |\tau| \big(a\|e^{-2\xi}\|_{L^\sigma(I_\tau)} + \|\mathcal{R}_1\|_{L^\sigma(I_\tau)} + \|G_1\|_{L^\sigma(I_\tau)}\big).\end{aligned}$$

Since $\xi = \frac{1}{2}\log(2b|t|) + h$ with $\|h\|^{\mu,1+\mu}_{1,\sigma,t_0} \le 1$, we have

$$|\tau|^\mu \|h\|_{L^\infty(I_\tau)} \le 1.$$

Hence, for $\tau < t_0 - 1$,

$$a|\tau|\|e^{-2\xi}\|_{L^\sigma(I_\tau)} \le \frac{a}{2b}\|e^{-2h}\|_{L^\infty(I_\tau)} \le \frac{a}{b}$$

provided that $|t_0|$ is chosen sufficiently large so that $\|e^{-2h}\|_{L^\infty(I_\tau)} \le 2$. Set

$$C_0 := \frac{2a}{Cb} \tag{5.16}$$

where C is the same constant as in (5.15) and constants a and b are defined in (5.3). This implies that C_0 is a universal constant depending only on the dimension n. We claim that

$$\|F_h\|^1_{\sigma,t_0} \le \frac{C_0}{2} + C|t_0|^{-\delta}, \quad \delta > 0. \tag{5.17}$$

To show this claim, we recall that by Corollary 5.1 we have

$$\sup_{\tau \le t_0 - 1} |\tau| \|\mathcal{R}_1\|_{L^\sigma(I_\tau)} \le C|t_0|^{-\gamma},$$

so we only need to show that

$$\sup_{\tau \le t_0 - 1} |\tau| \|G_1\|_{L^\sigma(I_\tau)} \le C|t_0|^{-\delta}$$

for some $\delta > 0$. To this end, we recall that

$$\|G_1\|_{L^\sigma(I_\tau)} = \left(\int_\tau^{\tau+1} \left(\int_{-\infty}^{\infty} Q(\psi) w_2^p \, dx\right)^\sigma dx\right)^{\frac{1}{\sigma}}$$

where $Q(\psi) := E(\psi) - z^{1-p}M$ is given in (2.6). To establish the above bound, we estimate term by term similarly as in the proof of Lemma 4.2. For example, for the term $z^{1-p}N_{13}$ which is given in (4.13) and satisfies estimate (4.15), if we also use that $w_2 \le z$, we have

$$\begin{aligned}&\left(\int_\tau^{\tau+1} \left(\int_{-\infty}^{\infty} N_{13} \frac{w_2^p}{z^{p-1}} \, dx\right)^\sigma dt\right)^{\frac{1}{\sigma}} \\ &\qquad \le C\left(\int_\tau^{\tau+1} \left(\int_{-\infty}^{\infty} |\psi|(|\psi_t| + |\dot\xi| + |\dot\eta||\psi|) w_2^{p-1} \, dx\right)^\sigma dt\right)^{\frac{1}{\sigma}}\end{aligned}$$

Recalling that $2\beta = p - 1$ and using the bounds

$$\|\psi\|_{L^\infty(\mathbb{R}\times[\tau,\tau+1])} \le |\tau|^{-\nu} \|\psi\|^\nu_{*,2,\sigma,t_0} \le 1, \quad w_2 \le z$$

together with Hölder's inequality, we obtain

$$
\begin{aligned}
&\Big(\int_\tau^{\tau+1}\Big(\int_{-\infty}^{\infty} N_{13}\frac{w_2^p}{z^{p-1}}\,dx\Big)^\sigma dt\Big)^{\frac{1}{\sigma}} \\
&\quad\le C\Big(\int_\tau^{\tau+1}\Big(\int_{|x|\ge\xi} |\psi||\psi_t| z^{\frac{n\beta-\sigma}{\sigma}} w_2^{p-\frac{n\beta}{\sigma}}\,dx\Big)^\sigma dt\Big)^{\frac{1}{\sigma}} \\
&\qquad + C\Big(\int_\tau^{\tau+1}\Big(\int_{|x|\le\xi} |\psi||\psi_t| z^{2\beta+\theta} w_2^{-\theta}\,dx\Big)^\sigma dt\Big)^{\frac{1}{\sigma}} + C|\tau|^{-1-\nu} \\
&\quad\le C\Big(\int_\tau^{\tau+1}\int_{-\infty}^{\infty}|\psi_t|^\sigma \alpha_\sigma\,dx\,dt\Big)^{\frac{1}{\sigma}}(|\tau|^{-\nu}+|\tau|^{-\nu+\theta}) + C|\tau|^{-1-\nu} \\
&\quad\le C|\tau|^{-2\nu+\theta} \\
&\quad\le C|\tau|^{-1-\delta}
\end{aligned}
$$

for some $\delta > 0$ and sufficiently small θ. Similar bounds hold for all the other terms which readily give the bound for $\|G_1\|_{L^\sigma(I_\tau)}$.

The above discussion establishes the bound (5.17). Using this bound, we finally obtain

$$\|A(\eta)\|^1_{1,\sigma,t_0} \le C\|F_h\|^1_{\sigma,t_0} \le \frac{C_0}{2} + C|t_0|^{-\delta} \le \frac{2}{3}C_0 < C_0$$

provided that $|t_0|$ is sufficiently large. We conclude that $A(K) \subset K$, where C_0 is the universal constant on the right-hand side of (5.17), finishing the proof of (a).

(b) There exists a universal constant $t_0 < 0$ for which $A : K \to K$ defines a contraction map. Since h is fixed, we only write in F_h, $\mathcal{R}_1$ and G_1 their dependence on ψ and η. As in part (a), for every $\eta^1, \eta^2 \in K$, if $\psi^i = \Psi(h, \eta^i)$, we have

$$
\begin{aligned}
\|A(\eta^1) - A(\eta^2)\|^1_{1,\sigma,t_0} &\le C\|F_h(\eta^1,\dot\eta^1,\psi^1) - F_h(\eta^2,\dot\eta^2,\psi^2)\|^1_{L^\sigma_{t_0}} \\
&\le C\big(\|\mathcal{R}_1(\eta^1,\dot{\eta^1}) - \mathcal{R}_1(\eta^2,\dot{\eta^2})\|^1_{L^\sigma_{t_0}} \\
&\qquad + \|G_1(\eta^1,\psi^1) - G_1(\eta^2,\psi^2)\|^1_{L^\sigma_{t_0}}\big).
\end{aligned} \tag{5.18}
$$

Observe first, using (5.9), that

$$
\begin{aligned}
|\mathcal{R}_1(\eta^1,\dot\eta^1) - \mathcal{R}_1(\eta^2,\dot\eta^2)| &\le C\Big|\int_{\eta^1}^{\eta^2}\partial_\eta\mathcal{R}_1(\eta,\dot\eta^1)\,d\eta\Big| + \Big|\int_{\dot\eta^1}^{\dot\eta^2}\partial_{\dot\eta}\mathcal{R}(\eta^2,\dot\eta)\,d\dot\eta\Big| \\
&\le C\Big(\int_{\eta^1}^{\eta^2} e^{-2\xi}(e^{-2\gamma\xi} + |\dot\xi| + |\eta| + |\dot\eta| + 1)\,d\eta \\
&\qquad + \int_{\dot\eta^1}^{\dot\eta^2}(e^{-2\xi} + |\eta^2|)\,d\dot\eta\Big)
\end{aligned}
$$

implying that

$$\|\mathcal{R}_1(\eta^1,\dot{\eta^1}) - \mathcal{R}_1(\eta^2,\dot{\eta^2})\|^1_{\sigma,t_0} \le \frac{C}{|t_0|}\|\eta^1-\eta^2\|^1_{1,\sigma,t_0}. \tag{5.19}$$

Furthermore, we claim

$$\begin{aligned}\|G_1(\eta^1,\psi^1)-G_1(\eta^2,\psi^2)\|^1_{\sigma,t_0} &\le \|G_1(\eta^1,\psi^1)-G_1(\eta^2,\psi^1)\|^1_{\sigma,t_0}\\ &\quad + \|G_1(\eta^2,\psi^1)-G_1(\eta^2,\psi^2)\|^1_{\sigma,t_0}\end{aligned}$$

where

$$\|G_1(\eta^1,\psi^1)-G_1(\eta^2,\psi^1)\|^1_{\sigma,t_0} \le C\,|t_0|^{-2\nu+\theta}\,\|\eta^1-\eta^2\|^1_{1,\sigma,t_0} \tag{5.20}$$

and

$$\begin{aligned}\|G_1(\eta^2,\psi^1)-G_1(\eta^2,\psi^2)\|^1_{\sigma,t_0} &\le C\,|t_0|^{1-2\nu+\theta}\,\|\psi^1-\psi^2\|^\nu_{*,2,\sigma,t_0}\\ &\le C\,|t_0|^{-2\nu+\theta}\,\|\eta^1-\eta^2\|^1_{1,\sigma,t_0}.\end{aligned} \tag{5.21}$$

To establish (5.20), as in part (a) let us look at the term coming from $z^{1-p}N_{13}(\psi)$. Using the estimate of this term obtained in the proof of Lemma 4.6 and a similar analysis as in part (a), we obtain

$$\begin{aligned}&\sup_{\tau\le t_0-1}|\tau|\bigg(\int_\tau^{\tau+1}\bigg(\int (N_{13}(\psi^1)(\eta^1)-N_{13}(\psi^1)(\eta^2))\frac{w_2^p}{z^{p-1}}\,dx\bigg)^\sigma dt\bigg)^{\frac{1}{\sigma}}\\ &\qquad \le C\,|t_0|^{-2\nu+\theta}\,\|\eta^1-\eta^2\|^1_{1,\sigma,t_0}\end{aligned}$$

where we used that $\|\psi^1\|^\nu_{*,2,\sigma,t_0}\le 1$ and that $(h,\eta_i)\in K$. All other terms in (5.20) can be estimated similarly, so estimate (5.20) follows. To establish (5.21) one argues similarly as above using the established bounds in the proof of Lemma 4.4 and in (4.6). Recall that $\nu>\frac{1}{2}$ and $\theta>0$ is small. Then combining (5.18)–(5.21) and taking $|t_0|$ sufficiently large yields the contraction bound

$$\|A(\eta^1)-A(\eta^2)\|^1_{1,\sigma,t_0} \le \frac{1}{2}\|\eta^1-\eta^2\|^1_{1,\sigma,t_0}. \tag{5.22}$$

This finishes the proof of part (b).

Having (a) and (b), we may apply the fixed point theorem to the operator $A: K\to K$ to conclude the existence of an $\eta=\eta(h)\in K$ so that $A(\eta)=\eta$.

(c) For any $h_1, h_2 \in K$, (5.13) holds. Since $A(\eta)=\eta$, we have

$$\eta(h)=\int_t^{t_0} e^{\lambda(t-s)}F_h(s)\,ds.$$

Hence, similarly as in the proof of (5.14),

$$\sup_{\tau\le t_0-1}|\tau|\,|\eta(h^1)-\eta(h^2)| \le C\,\|F_{h_1}-F_{h_2}\|^1_{2,t_0} \le C\,\|F_{h_1}-F_{h_2}\|^1_{\sigma,t_0}$$

which yields

$$\|\eta(h^1)-\eta(h^2)\|^1_{1,\sigma,t_0} \le C\,\|F_{h_1}-F_{h_2}\|^1_{\sigma,t_0}. \tag{5.23}$$

Recall that

$$F_h = ae^{-2\xi}+\mathcal{R}_1(\xi,\dot\xi,\eta,\dot\eta)+G_1(\psi,h,\eta)$$

and $\xi = \xi_0 + h$ with $\xi_0(t) = \frac{1}{2}\log(2b|t|)$ and $|h|$ small. Applying that to h^1 and h^2, we get

$$\begin{aligned}\|e^{-2(\xi_0+h^1)} - e^{-2(\xi_0+h^2)}\|^1_{\sigma,t_0} &\le C \sup_{\tau\le t_0-1}\left(\int_\tau^{\tau+1} |h^1-h^2|^\sigma\,dt\right)^{\frac{1}{\sigma}}\\ &\le |t_0|^{-\mu}\|h^1-h^2\|^{\mu}_{\sigma,t_0}.\end{aligned}$$

Next, using (5.9) and the notation $\eta^i = \eta(h^i) = \eta(\xi^i)$, where $\xi^i = \frac{1}{2}\log(2b|t|) + h^i$, we have

$$\begin{aligned}&\|\mathcal{R}_1(\xi^1,\dot\xi^1,\eta^1,\dot\eta^1) - \mathcal{R}_1(\xi^2,\dot\xi^2,\eta^2,\dot\eta^2)\|^1_{\sigma,t_0}\\ &\quad\le C\left\|\int_{\xi_1}^{\xi^2}\left|\frac{\partial\mathcal{R}_1}{\partial\xi}(\xi,\dot\xi^1,\eta^1,\dot\eta^1)\right|d\xi + \int_{\dot\xi^1}^{\dot\xi^2}\left|\frac{\partial\mathcal{R}_1}{\partial\dot\xi}(\xi^2,\dot\xi,\eta^1,\dot\eta^1)\right|d\dot\xi\right.\\ &\qquad\left.+\int_{\eta^1}^{\eta^2}\left|\frac{\partial\mathcal{R}_1}{\partial\eta}(\xi^2,\dot\xi^2,\eta,\dot\eta^1)\right|d\eta + \int_{\dot\eta^1}^{\dot\eta^2}\left|\frac{\partial\mathcal{R}_1}{\partial\dot\eta}(\xi^2,\dot\xi^2,\eta^2,\dot\eta)\right|d\dot\eta\right\|^1_{\sigma,t_0}\\ &\quad\le C\big(|t_0|^{-\mu}\|h^1-h^2\|^{\mu}_{\sigma,t_0} + |t_0|^{-1}\|\dot h^1-\dot h^2\|^{\mu+1}_{\sigma,t_0}\\ &\qquad + |t_0|^{-1}\|\eta^1-\eta^2\|^1_{\sigma,t_0} + |t_0|^{-1}\|\dot\eta^1-\dot\eta^2\|^1_{\sigma,t_0}\big)\\ &\quad\le C|t_0|^{-\delta}\big(\|h^1-h^2\|^{\mu,1+\mu}_{1,\sigma,t_0} + \|\eta(h^1)-\eta(h^2)\|^1_{1,\sigma,t_0}\big)\end{aligned}$$

if $\delta < \min\{1,\mu\}$. Finally, similarly to the estimates obtained in the proof of Proposition 5.2 and Lemma 4.6, if we denote for simplicity $\psi^i = \psi(h^i,\eta^i)$ with $\eta^i = \eta(h^i)$, using (5.20) and (5.21) we have

$$\begin{aligned}&\|G_1(\psi^1,h^1,\eta^1) - G_1(\psi^2,h^2,\eta^2)\|^1_{\sigma,t_0}\\ &\quad\le \|G_1(\psi^1,h^1,\eta^1) - G_1(\psi^1,h^2,\eta^1)\|^1_{\sigma,t_0} + \|G_1(\psi^1,h^2,\eta^1) - G_1(\psi^1,h^2,\eta^2)\|^1_{\sigma,t_0}\\ &\qquad + \|G_1(\psi^1,h^2,\eta^2) - G_1(\psi^2,h^2,\eta^2)\|^1_{\sigma,t_0}\\ &\quad\le C\big(|t_0|^{1-2\nu-\mu+\theta}\|h^1-h^2\|^{\mu,1+\mu}_{1,\sigma,t_0} + |t_0|^{-2\nu+\theta}\|\eta^1-\eta^2\|^1_{1,\sigma,t_0}\\ &\qquad + |t_0|^{1-2\nu+\theta}\|\psi^1-\psi^2\|^{\nu}_{*,2,\sigma,t_0}\big)\\ &\quad\le C\big(|t_0|^{1-2\nu-\mu+\theta}\|h^1-h^2\|^{\mu,1+\mu}_{1,\sigma,t_0} + |t_0|^{-2\nu+\theta}\|\eta^1-\eta^2\|^1_{1,\sigma,t_0}\\ &\qquad + |t_0|^{1-2\nu+\theta}\big(\|\psi(h^1,\eta(h^1)) - \psi(h^2,\eta(h^1))\|^{\nu}_{*,2,\sigma,t_0}\\ &\qquad\qquad + \|\psi(h^2,\eta(h^1)) - \psi(h^2,\eta(h^2))\|^{\nu}_{*,2,\sigma,t_0}\big)\big)\\ &\quad\le C\big(|t_0|^{1-2\nu-\mu+\theta}\|h^1-h^2\|^{\mu,1+\mu}_{1,\sigma,t_0} + |t_0|^{-\nu+\theta}\|\eta^1-\eta^2\|^1_{1,\sigma,t_0}\big)\end{aligned}$$

where we have used (4.5) and (4.6). For the above estimate we only need to check that

$$\|G_1(\psi^1,h^1,\eta^1) - G_1(\psi^1,h^2,\eta^1)\|^1_{\sigma,t_0} \le C|t_0|^{1-2\nu-\mu+\theta}\|h^1-h^2\|^{\mu,1+\mu}_{1,\sigma,t_0}.$$

Indeed, if we pick the term $z^{1-p}N_{13}$, using the estimates from the proof of Lemma 4.6 we have

$$\begin{aligned}&\sup_{\tau\le t_0}|\tau|\left(\int_\tau^{\tau+1}\left(\int_{-\infty}^{\infty}|N_{13}(\psi^1,\eta^1)(h^1) - N_{13}(\psi^1,\eta^1)(h^2)|w_2^p z^{1-p}\,dx\right)^\sigma dt\right)^{\frac{1}{\sigma}}\\ &\quad\le C\sup_{\tau\le t_0}|\tau|\left(\int_\tau^{\tau+1}\left(\int_{-\infty}^{\infty}|\psi||h^1-h^2|(|\psi_\tau| + |\psi|(|\dot\xi_1| + |\dot\xi_2| + |\dot\eta|))w_2^{p-1}\,dx\right)^\sigma dt\right)^{\frac{1}{\sigma}}\\ &\quad\le C|t_0|^{1-2\nu-\mu+\theta}\|h^1-h^2\|^{\mu,1+\mu}_{1,\sigma,t_0}.\end{aligned}$$

Combining the estimates from above yields

$$\|F_{h_1} - F_{h_2}\|^1_{2,t_0} \le C\big(|t_0|^{-\delta}\|h^1 - h^2\|^{\mu,1+\mu}_{1,\sigma,t_0} + |t_0|^{-\nu+\theta}\|\eta^1 - \eta^2\|^1_{1,\sigma,t_0}\big)$$

where $\delta < \min\{1, \mu, 1 - 2\nu - \mu + \theta\}$. Hence by (5.23) and choosing $|t_0|$ sufficiently large we conclude the bound (5.13), finishing the proof of (c). □

5.3. Solving for ξ. Recall that $\xi(t) = \xi_0(t) + h(t)$, $\xi_0(t) = \frac{1}{2}\log(2b|t|)$. Moreover, $\xi_0(t)$ is a solution to the homogeneous part of equation (5.8), namely

$$\dot{\xi}_0 + be^{-2\xi_0} = 0.$$

Using this last equation, we may rewrite equation (5.8) as follows:

$$\dot{h} + \frac{1}{t}h = F(h) \tag{5.24}$$

where

$$F(h) := G_3(h) + \mathcal{R}_2(\xi, \dot{\xi}, \eta, \dot{\eta}) + G_2(\psi, h, \eta)$$

with $G_2(\psi, h, \eta)$ as in Corollary 5.1 and

$$G_3(h) := -(\mathcal{G}(\xi_0 + h) - \mathcal{G}(\xi_0) - \mathcal{D}_\xi\mathcal{G}(\xi_0)h), \quad \mathcal{G}(\xi) = e^{-2\xi}. \tag{5.25}$$

Equivalently, the left-hand side of (5.24) is the linearization of equation (5.8) around ξ_0. We can in fact write (5.24) as a fixed point for a left inverse of that operator. The equation

$$\dot{h} + \frac{1}{t}h = g, \quad h(t_0) = 0 \tag{5.26}$$

is solved by the formula

$$h(t) := \mathcal{T}(g) = |t|^{-1}\int_t^{t_0} g(\tau)\tau\, d\tau \tag{5.27}$$

and then, writing

$$B(h) := \mathcal{T}(F(h)) = |t|^{-1}\int_t^{t_0} sF(h)(s)\, ds$$

it suffices to solve the fixed point problem $h = B(h)$ in a suitable region. Let us consider the set

$$K_0 = \big\{h : (-\infty, t_0] \to \mathbb{R} : \|h\|^{\mu,1+\mu}_{1,\sigma,t_0} \le 1\big\}.$$

We have the following result.

Proposition 5.3. *There exist a universal constant $t_0 < 0$ and function h, $\|h\|^{\mu,\mu+1}_{1,\sigma,t_0} \le 1$, so that $B(h) = h$.*

Proof. We will show that the map B leaves the sets invariant and that it is a contraction.

(a) There exists a universal constant $t_0 < 0$ so that $B(K_0) \subset K_0$. It is easy to see that if $t_0 < 0$ and $\sigma \ge 2$, we have

$$\sup_{\tau \le t_0 - 1} |\tau|^\mu |B(h)(\tau)| \le C\|F(h)\|^{1+\mu}_{2,t_0} \tag{5.28}$$

for a universal constant C. Denote by $I_\tau = [\tau, \tau+1]$. Using (5.28), we observe that

$$\begin{aligned}(5.29)\qquad \|B(h)\|^{\mu,\mu+1}_{1,\sigma,t_0} &= \sup_{\tau\le t_0-1} |\tau|^{\mu}\|B(h)\|_{L^\sigma(I_\tau)} + \sup_{\tau\le t_0-1}|\tau|^{1+\mu}\left\|\frac{d}{dt}B(h)\right\|_{L^\sigma(I_\tau)}\\ &\le C\|F(h)\|^{1+\mu}_{\sigma,t_0}.\end{aligned}$$

We will use this estimate to show that $B(K_0)\subset K_0$. We have

$$\begin{aligned}\|F(h)\|^{1+\mu}_{\sigma,t_0} &= \sup_{\tau\le t_0-1}|\tau|^{1+\mu}\|F(h)\|_{L^\sigma(I_\tau)}\\ &\le C\sup_{\tau\le t_0-1}|\tau|^{1+\mu}\big(\|\mathcal{R}_2\|_{L^\sigma(I_\tau)} + \|G_2\|_{L^\sigma(I_\tau)} + \|G_3\|_{L^\sigma(I_\tau)}\big).\end{aligned}$$

A straightforward computation, recalling (5.25) shows that

$$\|G_3(h)\|_{L^\sigma(I_\tau)} \le \frac{C}{|\tau|}\|h^2\|_{L^\sigma(I_\tau)}$$

implying (since $\|h\|^{\mu,1+\mu}_{1,\sigma,t_0}\le 1$) the bound

$$(5.30)\qquad \sup_{\tau\le t_0-1}|\tau|^{1+\mu}\|G_3(h)\|_{L^\sigma(I_\tau)} \le \frac{C}{|t_0|^{\mu}}.$$

By Corollary 5.1 we have

$$(5.31)\qquad \sup_{\tau\le t_0-1}|\tau|^{1+\mu}\|\mathcal{R}_2\|_{L^\sigma(I_\tau)} \le \frac{C}{|t_0|^{\gamma-\mu}}$$

where we choose μ so that $\mu<\gamma$. To establish the bound on $\sup_{\tau\le t_0}|\tau|^{1+\mu}\|G_2(h)\|_{L^\sigma(I_\tau)}$, we estimate term by term, similarly to the proof of Lemma 4.2. Similarly to deriving the estimate for the term G_1 in Section 5.2 we have

$$\|G_2\|_{L^\sigma(I_\tau)} \le C|\tau|^{-2\nu+\theta}$$

implying

$$\|G_2\|^{1+\mu}_{\sigma,t_0} \le C|\tau|^{-2\nu+\theta+1+\mu}$$

where we choose $\mu<\min\{\gamma, 2\nu-1\}$ and $\theta>0$ is chosen small so that $2\nu-\theta-1-\mu>0$. This together with (5.30) and (5.31) implies

$$\|B(h)\|^{\mu,1+\mu}_{1,\sigma,t_0} \le \|F(h)\|^{1+\mu}_{\sigma,t_0} \le \frac{C}{|t_0|^{\tilde\delta}},$$

where $\tilde\delta = \min\{\mu, \gamma-\mu, 2\nu-\theta-1-\mu\}$. We conclude that for $|t_0|$ sufficiently large we have $B(K_0)\subset K_0$.

(b) There exists a universal constant $t_0<0$ for which the map $B: K_0\to K_0$ defines a contraction map. Observe that similarly as in part (a) we have

$$(5.32)\qquad \|B(h^1)-B(h^2)\|^{\mu,\mu+1}_{1,\sigma,t_0} \le C\|F(h^1)-F(h^2)\|^{1+\mu}_{\sigma,t_0}.$$

An easy computation (the same as in part (a)) shows that

$$|G_3(h^1)-G_3(h^2)| \le \frac{C}{|t|}(|h^1-h^2|^2 + (|h^1|+|h^2|)|h^1-h^2|)$$

implying that

$$\|G_3(h^1) - G_3(h^2)\|_{\sigma,t_0}^{1+\mu} \le \frac{C}{|t_0|^\mu} \|h^1 - h^2\|_{1,\sigma,t_0}^{\mu,1+\mu}.$$

Let $\xi^i = \xi_0 + h^i$ and recall the notation $\eta^i = \eta(\xi^i)$. Then, similar to a discussion in part (c) of the proof of Proposition 5.2, we have

$$\begin{aligned}\|\mathcal{R}_2(\xi^1, \dot\xi^1, \eta(\xi^1), \dot\eta(\xi^1)) - \mathcal{R}_2(\xi^2, \dot\xi^2, \eta(\xi^2), \dot\eta(\xi^2))\|_{\sigma,t_0}^{1+\mu} \\ \le C|t_0|^{-\delta}\big(\|h^1 - h^2\|_{1,\sigma,t_0}^{\mu,1+\mu} + \|\eta(h^1) - \eta(h^2)\|_{1,\sigma,t_0}^1\big) \\ \le C|t_0|^{-\delta}\|h^1 - h^2\|_{1,\sigma,t_0}^{\mu,1+\mu}\end{aligned} \tag{5.33}$$

where in the last line we have used (5.13) where $\delta < \min\{1, \mu, 1 - 2\nu - \mu + \theta\}$. Furthermore, similar to deriving (5.13) in part (c) of the proof of Proposition 5.2, we get

$$\|G_2(\psi^1, h^1, \eta^1) - G_2(\psi^2, h^2, \eta^2)\|_{\sigma,t_0}^{\mu+1} \le C|t_0|^{1-2\nu+\theta}\|h^1 - h^2\|_{1,\sigma,t_0}^{\mu,1+\mu}.$$

Using (5.32), the definition of $F(h)$ and the estimates above, if we choose $|t_0|$ sufficiently large, we obtain the bound

$$\|B(h^1) - B(h^2)\|_{1,\sigma,t_0}^{\mu,1+\mu} \le \frac{1}{2}\|h^1 - h^2\|_{1,\sigma,t_0}^{\mu,1+\mu}$$

which finishes the proof of part (b).

By the fixed point theorem applied to the operator B, there exists an $h \in K_0$ so that $B(h) = h$, or in other words, h solves (5.24). This concludes the proof of the proposition. □

We conclude Section 5 by the proof of Proposition 5.1.

Proof of Proposition 5.1. By Proposition 5.2, for every $h \in K$, there exists an $\eta = \eta(h)$ so that (5.1) is satisfied. By Proposition 5.3, if we take $\eta = \eta(h)$, there exists an h, so that $\|h\|_{1,\sigma,t_0}^{\mu,\mu+1} \le 1$ and (5.2) holds. Take this pair of functions $(h, \eta) \in K$, for which both (5.1) and (5.2) are satisfied. Then by Proposition 4.1 there is a solution $\psi = \Psi(h, \eta)$ of (2.5), satisfying the orthogonality conditions (3.5) and (3.6). This finishes the proof of Proposition 5.1. □

6. Properties of the solution

Unlike the contracting spheres (1.4) and the King solution (1.5) to the Yamabe flow (1.1) which are both type I ancient solutions with positive Ricci curvature, the solution that we construct in Theorem 1.1 is of type II and its Ricci curvature changes its sign. More precisely, we have the following proposition.

Proposition 6.1. *The solution constructed in Theorem* 1.1 *is a type II ancient solution in the sense of Definition* 1.1. *Its Ricci curvature changes its sign.*

Proof. We recall that under the conformal change of the metric $g = e^{2f} g_{\mathbb{R}^n}$, the Ricci tensor changes as follows:

$$R_{ij} = -(n-2)(f_{ij} - f_i f_j) + (\Delta f - (n-2)|\nabla f|^2)\delta_{ij}.$$

In particular, if f is a radially symmetric function and we denote by R_{11} the Ricci tensor in the radial direction and by R_{jj}, with $j \geq 2$ the Ricci tensor in the spherical direction, then

$$R_{11} = -(n-3) f_{rr} + \frac{n-1}{r} f_r$$

and for $j \geq 2$,

$$R_{jj} = f_{rr} + \frac{n-1}{r} f_r - (n-2) f_r^2.$$

Observe that $R_{ij} = 0$ for $i \neq j$. We will next express R_{11} with respect to our conformal factor $u(x,t)$, expressed in cylindrical coordinates. Let $u(x,t)$ be the solution of (1.7) constructed in Theorem 1.1. Recall that $u(x,t)$ represents the conformal factor of the rescaled flow in cylindrical coordinates. Using the change of variables

$$f(r) = \frac{2}{n-2} \log u(x,\tau) - x, \quad r = e^x,$$

we find that R_{11} (the Ricci curvature of the rescaled metric in the radial direction) in terms of u is given by

$$R_{11} = -\frac{(n-2)^2 u^2 - (n-3) u_x^2 + u[(n-3)u_{xx} - 2(n-2)u_x]}{e^{2x} u^2}.$$

The Ricci curvature changes its sign. We will show that R_{11} changes sign. Note that the sign of R_{11} is determined by the sign of

$$Q := -((n-2)^2 u^2 - (n-3) u_x^2 + u[(n-3)u_{xx} - 2(n-2)u_x]).$$

Recall that our solution u is given by

$$u = (1+\eta)z + \psi = z + \tilde{\psi}, \quad \tilde{\psi} := \eta z + \psi.$$

Let $Q := Q_1 + Q_2$, where

$$Q_1 := (n-3) z_x^2 - (n-2)^2 z^2 - z[(n-3) z_{xx} - 2(n-2) z_x]$$

and Q_2 is the error term that is a linear combination of $\tilde{\psi}_x^2$, $\tilde{\psi}_x z_x$, $\tilde{\psi}^2$, $\tilde{\psi} z$, $\tilde{\psi}\tilde{\psi}_{xx}$, $\tilde{\psi} z_{xx}$, $\tilde{\psi} z_x$, $\tilde{\psi}_{xx} z$. An easy computation shows that for $\tau < t_0 - 1$,

$$\int_\tau^{\tau+1} \int_{-1}^{1} Q_1 z^{p-1} \, dx \, dt \sim -\frac{C(n)}{|\tau|^{\frac{p+1}{2}}} < 0, \quad \text{as } \tau \to -\infty. \tag{6.1}$$

Hence, it will be sufficient to show that

$$\int_\tau^{\tau+1} \int_{-1}^{1} Q_2 z^{p-1} \, dx \, dt = o(|\tau|^{-\frac{p+1}{2}}), \quad \text{as } \tau \to -\infty.$$

Let us check that is the case for some of the terms that enter in the expression for Q_2. All the other terms can be checked similarly. To simplify the notation, set $R_\tau := [-1,1] \times [\tau, \tau+1]$. Using the energy estimate (3.23) and the fact that

$$z^{p-1} \leq C|\tau|^{-\frac{p-1}{2}} \quad \text{on } R_\tau,$$

we obtain the bound

$$\begin{aligned}(6.2)\qquad \iint_{R_\tau} (\tilde{\psi}^2 + \tilde{\psi}_x^2) z^{p-1}\,dx\,dt &\\ &\le |\tau|^{-\frac{p-1}{2}} \left((\|\eta\|^1_{\infty,t_0})^2 \iint_{R_\tau} z^2\,dx\,dt + \iint_{R_\tau} (\psi^2 + \psi_x^2)\,dx\,dt \right)\\ &\le |\tau|^{-\frac{p-1}{2}} (|\tau|^{-2} + |\tau|^{-2\nu})\\ &\le C|\tau|^{-\frac{p+3}{2}}\end{aligned}$$

and also

$$\begin{aligned}(6.3)\qquad \iint_{R_\tau} \tilde{\psi}\tilde{\psi}_{xx} z^{p-1}\,dx\,dt &\le C|\tau|^{-\frac{p-1}{2}} \left((\|\eta\|^1_{\infty,t_0})^2 \iint_{R_\tau} z^2\,dx\,dt \right.\\ &\qquad \left. + \|\eta\|^1_{\infty,t_0} \iint_{R_\tau} z\psi_{xx} + \psi z_{xx}\,dx\,dt \right)\\ &\qquad + \iint_{R_\tau} \psi\psi_{xx} z^{p-1}\,dx\,dt\\ &\le C|\tau|^{-\frac{p+2}{2}-\nu} + \iint_{R_\tau} \psi\psi_{xx} z^{p-1}\,dx\,dt.\end{aligned}$$

For the last term in (6.3), using the energy estimates (3.23) again, we have

$$\begin{aligned}(6.4)\qquad \iint_{R_\tau} \psi\psi_{xx} z^{p-1}\,dx\,dt &\le \left(\iint_{R_\tau} \psi^2 z^{p-1}\,dx\,dt \right)^{\frac12} \left(\iint_{R_\tau} \psi_{xx}^2 z^{p-1}\,dx\,dt \right)^{\frac12}\\ &\le C\|\psi\|_{L^\infty(\Lambda_\tau)} |\tau|^{-\frac{p-1}{2}} \left(\iint_{R_\tau} \psi_{xx}^2\,dx\,dt \right)^{\frac12}\\ &\le C|\tau|^{-\frac{p-1}{2}-2\nu}.\end{aligned}$$

We see that $\frac{p-1}{2} + 2\nu > \frac{p+1}{2}$ is equivalent to $2\nu > 1$, which is true. Using (3.23), we have

$$\begin{aligned}(6.5)\qquad \iint_{R_\tau} \tilde{\psi}_{xx} z^p\,dx\,dt &\le C\|\eta\|^1_{\infty,t_0} |\tau|^{-\frac{p+1}{2}} + \iint_{R_\tau} \psi_{xx} z^p\,dx\,dt\\ &\le C\big(|\tau|^{-\frac{p+3}{2}} + |\tau|^{-\frac{p}{2}-\nu}\big).\end{aligned}$$

Combining (6.1), (6.2), (6.3), (6.4), (6.5) yields

$$\iint_{R_\tau} Q z^{p-1}\,dx\,dt \le -C(n)|\tau|^{-\frac{p+1}{2}}$$

if τ is sufficiently close to $-\infty$. Hence R_{11} has to be negative somewhere on R_τ if τ is sufficiently close to $-\infty$. On the other hand, since the scalar curvature R of our ancient metric is positive, the Ricci curvature must be positive somewhere. The conclusion is that the Ricci curvature of our ancient solution changes its sign all the way to $t \to -\infty$.

The solution is of type II. We observe that our rescaled ancient solution is of type II if

$$\limsup_{t\to-\infty} |\mathrm{Rm}|(\cdot, t) = +\infty.$$

Hence, it will be sufficient to show that

$$\limsup_{t\to-\infty} |\mathrm{Ric}|(\cdot, t) = +\infty. \tag{6.6}$$

Since, as we have noticed above, the Ricci curvature of our radially symmetric solution is diagonal, we have

$$|\mathrm{Ric}|^2 = R_{11}^2(g^{11})^2 + \sum_{i\geq 2} R_{ii}^2(g^{ii})^2.$$

Look at the $R_{11}g^{11} = R_{11}u^{-\frac{4}{n-2}}$, which is

$$R_{11}g^{11} = \frac{(n-3)u_x^2 - (n-2)^2u^2 - u[(n-3)u_{xx} - 2(n-2)u_x]}{e^{2x}u^{\frac{2n}{n-2}}}.$$

As in part (a), write $u = z + \tilde{\psi}$. Since $\|\psi\|_{L^\infty(\Lambda_\tau)} \leq C|\tau|^{-\nu}$, and $z \sim \tilde{C}|\tau|^{-\frac{1}{2}}$ on R_τ, we have $\frac{z}{2} \leq u \leq 2z$ on R_τ. Let us write

$$R_{11}g^{11} = J_1 + J_2$$

where the term J_1 we obtain from $R_{11}g^{11}$, after replacing u by z, and J_2 is the difference of those two terms. An easy computation shows that

$$\left|\iint_{R_\tau} J_1 z^{p-1}\,dx\,dt\right| \sim C|\tau|^{\frac{2}{n-2}} \iint_{R_\tau} z^{p-1}\,dx\,dt \sim C, \quad \text{as } \tau\to-\infty. \tag{6.7}$$

Using the energy estimate (3.23), the fact that $z \sim |\tau|^{-\frac{1}{2}}$ on R_τ, very similar estimates to those in part (a) show that

$$\left|\iint_{R_\tau} J_2 z^{p-1}\,dx\,dt\right| \leq C|\tau|^{-q} \tag{6.8}$$

for some $q > 0$. Combining (6.7) and (6.8), we see that

$$\left|\iint_{R_\tau} R_{11}g^{11}z^{p-1}\,dx\,dt\right| \geq c > 0$$

for all τ sufficiently close to $-\infty$. Since $z^{p-1} \sim |\tau|^{-\frac{2}{n-2}}$ on R_τ, the last estimate implies there exists a uniform constant $\delta > 0$ so that for every $\tau \leq \tau_0$, with τ_0 sufficiently close to $-\infty$, there exists an $(x_\tau, \tau) \in R_\tau$ so that

$$R_{11}g^{11}(x_\tau, \tau) \geq \delta|\tau|^{\frac{2}{n-2}}.$$

We conclude that (6.6) holds and our solution is of type II. □

We conclude this section with a final remark on our shape of our solution, as $t \to -\infty$.

Remark 6.1. The ancient solution $u(x, t)$ constructed in Theorem 1.1 looks like a tower of two bubbles as $t \to -\infty$.

More precisely, for any $\delta \in (0, 1)$ it is easy to check that we have the following:

(a) For $x < \xi(t)(1-\delta)$ we have $|u(x,t) - w(x+\xi(t))| < C|t|^{-\frac{\delta}{2}}$, which means that in this considered region we are close to one of the spheres (bubbles).

(b) For $x > -\xi(t)(1-\delta)$ we have $|u(x,t) - w(x-\xi(t))| < C|t|^{-\frac{\delta}{2}}$, which means that in this considered region we are close to the other sphere (bubble).

(c) For any $x_1, x_2 \in [-\xi(t)(1-\delta), \xi(t)(1-\delta)]$ and any two corresponding points $p_1, p_2 \in \mathbb{R}^n$ whose radial variables correspond to x_1, x_2 in cylindrical coordinates respectively, we have that

$$\operatorname{dist}_{g(t)}(p_1, p_2) \le C|t|^{-\frac{\delta}{n-2}} \log|t|,$$

which means that in the region where the two spheres interfere we have a short and narrow neck connecting the two bubbles and as $t \to -\infty$ this neck becomes shorter and narrower. We denoted by $g(\cdot)$ the rescaled metric.

To check (c) observe that

$$\begin{aligned}\operatorname{dist}_{g(t)}(p_1, p_2) &\le \int_{-\xi(t)(1-\delta)}^{\xi(t)(1-\delta)} (w(x-\xi(t)) + w(x+\xi(t)) + \psi(x,t))^{\frac{2}{n-2}}\, dx \\ &\le C\left(|t|^{-\frac{\delta}{2}} + |t|^{-\nu}\right)^{\frac{2}{n-2}} \log|t| \\ &\le C|t|^{-\frac{\delta}{n-2}} \log|t|.\end{aligned}$$

We finish the paper with the proof of the Main Theorem 1.1, which we restate below.

Theorem 6.1. *Let $p := (n+2)/(n-2)$ with $n \ge 3$. There exist a constant $t_0 = t_0(n)$ and a radially symmetric solution $u(x,t)$ to* (1.7) *defined on $\mathbb{R}\times(-\infty, t_0]$ of the form* (1.11)–(1.12)*, where the functions $\psi := \psi(x,t)$, $\xi := \frac{1}{2}\log(2b|t|) + h(t)$ and $\eta := \eta(t)$ satisfy*

$$\|\psi\|^{\nu}_{*,2,\sigma,t_0} < \infty, \qquad \|h\|^{\mu,1+\mu}_{1,\sigma,t_0} < \infty, \qquad \|\eta\|^{\mu,}_{1,\sigma,t_0} < \infty$$

(according to Definitions 2.6 *and* 2.7*). The constants σ, μ, ν and b are all positive and depend only on the dimension n.*

It follows that the solution u defines a radially symmetric ancient solution to the Yamabe flow (1.1) *on S^n which is of type II (in the sense of Definition* 1.1*) and its Ricci curvature changes its sign.*

Proof. Proposition 5.1 gives us, for $|t_0|$ sufficiently large, the existence of a radially symmetric solution ψ to (2.5) on $\mathbb{R}\times(-\infty, t_0]$, which is equivalent to the existence of a radially symmetric solution u to (1.7), defined on $\mathbb{R}\times(-\infty, t_0]$. This finishes the proof of the first part, that is, the existence part of Theorem 1.1. Furthermore, by Proposition 6.1 our constructed solution is a type II ancient solution to the Yamabe flow, with the Ricci curvature that changes its sign. □

7. The case of multiple bubbles

In this section we sketch the proof of Theorem 1.2, indicating the changes needed with respect to the two-bubble case. Now we seek for a solution of (1.7) of the form

$$u(x,t) = \sum_{j=1}^{k} (1 + \eta_j(t)) w_j(x,t) + \psi(x,t)$$

where $w_j(x,t) = w(x - \xi_j(t))$ and $\psi(x,t) \to 0$ as $t \to -\infty$. The functions $\xi_j(t)$ are ordered and symmetric,

$$\xi_1(t) < \xi_2(t) < \cdots < \xi_k(t), \quad \xi_j(t) = -\xi_{k-j+1}(t). \tag{7.1}$$

We also assume $\eta_j(t) = \eta_{k-j+1}(t)$.

We let $\xi(t) = (\xi_1(t), \dots, \xi_k(t))$. Let us write

$$\xi(t) = \xi_0(t) + h(t) \tag{7.2}$$

where $\xi_0(t)$ is the solution of the first order Toda system (1.16) as in the statement of the theorem that we will analyze later. The parameter functions $h_j(t)$ and $\eta_j(t)$ will decay in $|t|$, as $|t| \to \infty$ and will be chosen according to a system of first order differential equations.

We write now and

$$\tilde{z} := \sum_{j=1}^{k} (1 + \eta_j) w_j, \quad z := \sum_{j=1}^{k} w_j.$$

Equation (1.7) then becomes

$$\partial_t (\tilde{z} + \psi)^p = (\partial_x^2 \psi - \psi + \partial_x^2 \tilde{z} - \tilde{z}) + (\tilde{z} + \psi)^p.$$

Using that $\partial_x^2 w_j - w_j = -w_j^p$, we obtain the equation

$$\partial_t (\tilde{z} + \psi)^p = \left(\partial_x^2 \psi - \psi - \sum_{j=1}^{k} (1 + \eta_j) w_j^p \right) + (\tilde{z} + \psi)^p$$

which can be re-written as

$$p z^{p-1} \partial_t \psi = \partial_{xx} \psi - \psi + p z^{p-1} \psi - z^{p-1} C(\psi, t) + z^{p-1} E(\psi) \tag{7.3}$$

where now

$$C(\psi) = \sum_{j=1}^{k} d_{1j}(t) w_j + d_{2j}(t) w_j'$$

and the coefficients d_{1j}, d_{2j} are precisely chosen as linear functions of ψ in such a way that the following (nearly diagonal) system holds:

$$\begin{aligned}
&\int_{\mathbb{R}} z^{1-p} (\partial_{xx} \psi - \psi + p z^{p-1} \psi - p z^{p-1} \partial_t \psi) w_l^{p-1} w_l' \, dx \\
&\qquad = \sum_j d_{1j} \int_{\mathbb{R}} w_l^{p-1} w_j' w_l' \, dx + d_{2j} \int_{\mathbb{R}} w_l^{p-1} w_j w_l' \, dx, \\
&\int_{\mathbb{R}} z^{1-p} (\partial_{xx} \psi - \psi + p z^{p-1} \psi - p z^{p-1} \partial_t \psi) w_l^{p-1} w_l \, dx \\
&\qquad = \sum_j d_{1j} \int_{\mathbb{R}} w_l^{p-1} w_j' w_l \, dx + d_{2j} \int_{\mathbb{R}} w_l^{p-1} w_l w_j \, dx,
\end{aligned}$$

and $E(\psi)$ is given by

$$E(\psi) := z^{1-p} M + C(\psi) + z^{1-p} \big[(1 - \partial_t) N(\psi) - p \psi \partial_t z^{p-1} \big] \tag{7.4}$$

where

$$M := \tilde{z}^p - \sum_{j=1}^{k} (1 + \eta_j(t)) w_j^p - \partial_t \tilde{z}^p \tag{7.5}$$

and

$$N(\psi) := (\tilde{z} + \psi)^p - \tilde{z}^p - p\tilde{z}^{p-1}\psi + p\psi(\tilde{z}^{p-1} - z^{p-1}). \tag{7.6}$$

We consider first the auxiliary equation

$$\begin{aligned} pz^{p-1}\partial_t \psi = \partial_{xx}\psi - \psi + pz^{p-1}\psi - z^{p-1}C(\psi) \\ + z^{p-1}\Big[E(\psi) - \sum_j (c_{1j}(t)w_j' + c_{2j}(t)w_j)\Big] \end{aligned} \tag{7.7}$$

under the orthogonality conditions

$$\int_{\mathbb{R}} \psi w_j' w_j^{p-1}\, dx = 0 \quad \text{for a.e. } t < t_0 \tag{7.8}$$

and

$$\int_{\mathbb{R}} \psi w_j w_j^{p-1}\, dx = 0 \quad \text{for a.e. } t < t_0 \tag{7.9}$$

where $c_{1j}(t)$ and $c_{2j}(t)$ are uniquely chosen so that the function

$$\bar{E} := E(\psi) - \sum_{j=1}^{k} c_{1j}(t)w_j' + c_{2j}(t)w_j \tag{7.10}$$

satisfies the orthogonality conditions (7.8)–(7.9).

Following the same arguments of the case $k = 2$, only at the expense of further notation, we find for each given vectors h and η a solution $\psi = \Psi(h, \eta)$ to problem (7.7) subject to the orthogonality conditions (7.8)–(7.9) for ψ, in a similar way to that found in Proposition 4.1. After this, the problem is thus reduced to choosing the parameter vector functions $h(t)$ and $\eta(t)$ so that $c_{lj}(t) \equiv 0$, where h and η decay in t, as $t \to -\infty$. The procedure is exactly the same as before. Now we need to solve the system

$$\int_{\mathbb{R}} w_j^{p-1} w_j' E(\Psi(h,\eta))\, dx = \int_{\mathbb{R}} w_j^{p-1} w_j E(\Psi(h,\eta))\, dx = 0 \quad \text{for all } j = 1, \dots, k.$$

Similar computations to those in Section 5 yield that the above system has the form

$$0 = \dot{\eta}_j - \frac{p-1}{p}\eta_j - a\Big(e^{-(\xi_{j+1}-\xi_j)} - e^{-(\xi_j - \xi_{j-1})}\Big) + \mathcal{Q}_1(\xi, \dot{\xi}, \eta, \dot{\eta})$$

and

$$0 = b^{-1}\dot{\xi}_j + e^{-(\xi_{j+1}-\xi_j)} - e^{-(\xi_j-\xi_{j-1})} + \mathcal{Q}_2(\xi, \dot{\xi}, \eta, \dot{\eta})$$

where $\xi_0 \equiv -\infty$, $\xi_{k+1} \equiv +\infty$ and the terms $\mathcal{Q}_l$, $l = 1, 2$, decay as suitable negative powers of $|t|$ provided that the terms η and h do so (recall that $\xi_j(t) = \xi_{0j}(t) + h_j(t)$). We proceed in a similar way as in Section 5, solving first for vector function η as a function of h using a fixed point formulation as that in Section 5.2. The problem gets at last reduced to solving a perturbation of the Toda system

$$b^{-1}\dot{\xi}_j(t) + e^{-(\xi_{j+1}-\xi_j)} - e^{-(\xi_j-\xi_{j-1})} = 0, \quad j = 1, \dots, k,\ t \in (-\infty, -t_0]. \tag{7.11}$$

Let us find first the explicit solution ξ_0. Let us set

$$R_\ell(\xi) := -\Big[e^{-(\xi_\ell-\xi_{\ell-1})} - e^{-(\xi_{\ell+1}-\xi_\ell)}\Big] \tag{7.12}$$

and

$$\mathbf{R}(\xi) := \begin{bmatrix} R_1(\xi) \\ \vdots \\ R_m(\xi) \end{bmatrix}. \tag{7.13}$$

We want to solve the system $\dot{\xi} + \mathbf{R}(\xi) = 0$. To do so, we find first a convenient representation of the operator $\mathbf{R}(\xi)$. Let us consider the auxiliary variables

$$\mathbf{v} := \begin{bmatrix} \bar{\mathbf{v}} \\ v_k \end{bmatrix}, \quad \bar{\mathbf{v}} = \begin{bmatrix} v_1 \\ \vdots \\ v_{k-1} \end{bmatrix},$$

defined in terms of ξ as

$$v_\ell = \xi_{\ell+1} - \xi_\ell \quad \text{with } \ell = 1, \dots, k-1, \quad v_k = \sum_{\ell=1}^{k} \xi_\ell,$$

with the conventions $v_0 = v_{k+1} = +\infty$ and define the operators

$$\mathbf{S}(\mathbf{v}) := \begin{bmatrix} \bar{\mathbf{S}}(\bar{\mathbf{v}}) \\ 0 \end{bmatrix}, \quad \bar{\mathbf{S}}(\bar{\mathbf{v}}) = \begin{bmatrix} S_1(\bar{\mathbf{v}}) \\ \vdots \\ S_{k-1}(\bar{\mathbf{v}}) \end{bmatrix}$$

where

$$S_\ell(\mathrm{v}) := R_{\ell+1}(\xi) - R_\ell(\xi) = \begin{cases} e^{-v_2} - 2e^{-v_1} & \text{if } \ell = 1, \\ e^{-v_{\ell+1}} - 2e^{-v_\ell} + e^{-v_{\ell-1}} & \text{if } 1 < \ell < k-1, \\ -2e^{-v_{k-1}} + e^{-v_{k-2}} & \text{if } \ell = k-1. \end{cases}$$

Then the operators $\mathbf{R}$ and $\mathbf{S}$ are in correspondence through the formula

$$\mathbf{S}(\mathrm{v}) \;=\; \mathbf{B}\mathbf{R}(\mathbf{B}^{-1}\mathrm{v}) \tag{7.14}$$

where $\mathbf{B}$ is the constant, invertible $N \times N$ matrix

$$\mathbf{B} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}, \tag{7.15}$$

and then through the relation $\xi = \mathbf{B}^{-1}\mathrm{v}$ the system $\dot{\xi} + \mathbf{R}(\xi) = 0$ is equivalent to $\dot{\mathbf{v}} + \mathbf{S}(\mathbf{v}) = 0$ which decouples into

$$\dot{\bar{\mathbf{v}}} + \bar{\mathbf{S}}(\bar{\mathbf{v}}) = 0, \tag{7.16}$$

$$\dot{v}_k = 0 \tag{7.17}$$

where

$$\bar{\mathbf{S}}(\bar{\mathbf{v}}) := \mathbf{C}\begin{bmatrix} e^{-v_1} \\ \vdots \\ e^{-v_{k-1}} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & & -1 & 2 \end{bmatrix}. \tag{7.18}$$

We choose simply $v_k = 0$ and look for a solution $\mathbf{v}^0(t) = (\bar{\mathbf{v}}^0(t), 0)$ of the system, where $\bar{\mathbf{v}}^0(t)$ has the form

$$\mathbf{v}^0_\ell(t) = \log(-t) + \beta_\ell$$

for constants β_ℓ to be determined.

Substituting this expression into the system, we find the following equations for the numbers β_ℓ:

$$\mathbf{C}\begin{bmatrix} e^{-\beta_1} \\ \vdots \\ e^{-\beta_{m-1}} \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

We compute explicitly

$$\beta_\ell = \log\left[\frac{1}{2\alpha}(k-\ell)\ell\right], \quad \ell = 1, \dots, k-1, \tag{7.19}$$

and hence we find a solution $\xi^0(t) = \mathbf{B}^{-1}\mathrm{v}^0$ to system (1.16), which is explicitly given by

$$\xi^0_j(t) = \left(j - \frac{k+1}{2}\right)\log(-t) + \gamma_j, \quad j = 1, \dots, k,$$

where

$$\gamma_1 = -\frac{1}{k}\sum_{j=1}^{k-1}\sum_{l=1}^{j}\beta_l, \quad \gamma_j = \gamma_1 + \sum_{l=1}^{j-1}\beta_l, \quad 2 \le j \le k.$$

This solution of the system is symmetric, in the sense that

$$\xi^0_{k-j+1} = -\xi^0_j, \quad j \le \frac{k}{2}.$$

After ξ_0 has been built, we follow exactly the scheme of Section 5.3. We linearize around ξ_0 and the equation for $\xi = \xi_0 + h$ takes the form, analogous to (5.24),

$$\dot{h} + D_\xi\mathbf{R}(\xi^0)h = F(h) \quad \text{in } (-\infty, -t_0) \tag{7.20}$$

where the nonlinear term $F(h)$ is small with h and it roughly has a decay at least one negative power more of $|t|$ than that of h. At this point we make the following important observation.

Since we are restricting ourselves to symmetric ξ, it follows that h satisfies the symmetry condition

$$h_{k-j+1} = -h_j, \quad j \le \frac{k}{2},$$

and the solution ψ is even in x, then examining closely the symmetries of $F(h)$ we find that

$$F(h)_{k-j+1} = -F(h)_j.$$

An exact analog of Proposition 5.3 does indeed hold, for a suitable inverse of the linear operator of the left-hand side of (7.21) in a subset of the vector functions h satisfying the symmetry requirement, which we construct next. Thus, we are reduced to building a solution of the linear system

$$\dot{h} + D_\xi \mathbf{R}(\xi^0)h = \mathbf{g} \quad \text{in } (-\infty, -t_0) \tag{7.21}$$

where $\mathbf{g}$ is such that $\mathbf{g}_{k-j+1} = -\mathbf{g}_j$. We observe that, in particular,

$$\sum_{j=1}^{k} \mathbf{g}_j \equiv 0.$$

We observe that

$$\mathbf{B}^{-1} D_{\mathbf{v}} \mathbf{S}(\mathbf{v}^0)\mathbf{B} = D_\xi \mathbf{R}(\xi^0),$$

hence setting

$$\mathtt{p} = \mathbf{B}h, \quad \mathtt{q} = \mathbf{B}\mathbf{g},$$

system (7.21) becomes equivalent to

$$\dot{\mathtt{p}} + D_{\mathbf{v}} \mathbf{S}(\mathbf{v}^0)\mathtt{p} = \mathtt{q} \quad \text{in } (-\infty, t_0). \tag{7.22}$$

Writing $\mathtt{p} = (\bar{\mathtt{p}}, \mathtt{p}_k)$, $\mathtt{q} = (\bar{\mathtt{q}}, \mathtt{q}_k)$, the latter system decouples as

$$\dot{\bar{\mathtt{p}}} + \mathtt{D}_{\bar{\mathbf{v}}} \bar{\mathbf{S}}(\bar{\mathbf{v}}^0)\bar{\mathtt{p}} = \bar{\mathtt{q}} \quad \text{in } (-\infty, t_0), \tag{7.23}$$

$$\dot{\mathtt{p}}_k = \mathtt{q}_k \quad \text{in } (-\infty, t_0). \tag{7.24}$$

We keep in mind that

$$\mathtt{q}_k = \sum_{j=1}^{k} \mathbf{g}_j = 0.$$

We simply choose $p_k(t) = 0$. Now,

$$D\bar{\mathbf{S}}(\bar{\mathbf{v}}) = -\mathbf{C} \begin{bmatrix} e^{-v_1} & 0 & \cdots & 0 \\ 0 & e^{-v_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-v_{k-1}} \end{bmatrix} \tag{7.25}$$

$$= \frac{1}{t} \begin{bmatrix} 2a_1 & -a_2 & 0 & & \cdots & 0 \\ -a_1 & 2a_2 & -a_3 & & \cdots & 0 \\ 0 & -a_2 & 2a_3 & & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \cdots & -a_{m-3} & 2a_{m-2} & -a_{m-1} \\ 0 & & \cdots & & -a_{m-2} & 2a_{m-1} \end{bmatrix},$$

with

$$a_\ell = \frac{\alpha^{-1}}{2}(k-\ell)\ell, \quad \ell = 1, \dots, k-1. \tag{7.26}$$

We shall solve problem (7.23). It is convenient to express it replacing the matrix $D_{\bar{\mathbf{v}}}\bar{\mathbf{S}}(\bar{\mathbf{v}}^0)$ with a symmetric one. We recall that we have

$$D_{\bar{\mathbf{v}}}\bar{\mathbf{S}}(\bar{\mathbf{v}}^0) = \frac{1}{t}\mathbf{C}\begin{bmatrix} e^{-\beta_1} & 0 & \cdots & 0 \\ 0 & e^{-\beta_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\beta_{k-1}} \end{bmatrix}$$

where the matrix $\mathbf{C}$ is given in (7.18); $\mathbf{C}$ is symmetric and positive definite. Indeed, a straightforward computation yields that its eigenvalues are explicitly given by

$$1, \frac{1}{2}, \dots, \frac{k-1}{k}.$$

We consider the symmetric, positive definite square root matrix of $\mathbf{C}$ and denote it by $\mathbf{C}^{\frac{1}{2}}$. Then setting

$$\bar{\mathrm{p}} := \mathbf{C}^{\frac{1}{2}}\omega, \quad \tilde{q} := \mathbf{C}^{-\frac{1}{2}}\bar{\mathrm{q}},$$

we see that equation (7.23) becomes

$$\dot{\omega} + \frac{1}{t}\mathbf{A}\omega = \tilde{q} \quad \text{in } (-\infty, t_0) \tag{7.27}$$

where $\mathbf{A}$ is the symmetric matrix

$$\mathbf{A} = \mathbf{C}^{\frac{1}{2}}\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m-1} \end{bmatrix}\mathbf{C}^{\frac{1}{2}} \tag{7.28}$$

where

$$a_\ell = \frac{1}{2}\ell(m-\ell).$$

In particular, $\mathbf{A}$ has positive eigenvalues. Let $u_1, \dots, u_{k-1}$ be an orthonormal set of eigenvectors of $\mathbf{A}$ associated to its positive eigenvalues $\lambda_1, \dots, \lambda_{k-1}$. Then writing

$$\omega(t) = \sum_{\ell=1}^{k-1}\omega_\ell(t)u_\ell, \quad \tilde{q}(t) = \sum_{\ell=1}^{k-1}\tilde{q}_\ell(t)u_\ell$$

we arrive at the equations

$$\dot{\omega}_\ell + \frac{\lambda_\ell}{t}\omega_\ell = \tilde{q}_\ell(t)$$

or

$$[(-t)^{\lambda_\ell}\omega_\ell]' = (-t)^{\lambda_\ell}\tilde{q}_\ell(t)$$

and then we find a solution by setting

$$\omega_\ell(t) = -(-t)^{-\lambda_\ell} \int_t^{t_0} (-\tau)^{\lambda_\ell} \tilde{q}_\ell(\tau)\, d\tau.$$

In this discussion, we have thus built an inverse

$$h = \mathcal{T}[\mathbf{g}]$$

to system (7.21) which is precisely its unique solution with $h(t_0) = 0$. This inverse clearly satisfies the symmetry condition

$$h_{k-j+1} = -h_j \quad \text{for } j \leq \frac{k}{2}$$

in case $\mathbf{g}$ does. The main fact to be observed is that if $\mathbf{g} = O(|t|^{-1-\mu})$ with $0 < \mu < 1$, then

$$h = O(|t|^{-\mu})$$

with bounded control in the sense of the norms used in Section 5.3. The remaining of the proof of existence of a solution to the nonlinear problem (7.20) goes essentially the same as in Section 5.3. □

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